## Math 447 Spring 2015 <br> Exam 3

April 21, 2015

- Total value 220 points. Each part valued as indicated.
- SHOW YOUR WORK unless otherwise indicated. "NO WORK" may result in "NO POINTS".
- Simplify your answers when possible. Do the arithmetic, remove parentheses, reduce fraction, etc.
- Cross out anything you don't want graded!
- Use the back sides of pages if you need extra space. If you have anything on a back side that you want graded, indicate where it is.

Student: $\qquad$
Section (please circle): 01 (Xu)

| Problem \# | Possible Points | Points |
| :---: | :---: | :---: |
| I | 30 |  |
| II | 14 |  |
| III | 21 |  |
| IV | 36 |  |
| V | 30 |  |
| VI | 28 |  |
| VII | 28 |  |
| VIII | 33 |  |
| Total | 220 |  |

I. (30 points.) $X$ has probability mass function (PMF; i.e., $p(x)=P\{X=x\}$ ) given in the table below

| $x$ | -2 | 0 | 2 | otherwise |
| :---: | :---: | :---: | :---: | :---: |
| $p(x)=P\{X=x\}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{1}{6}$ | 0 |

(1) (8 points) Give the distribution function (CDF) of $X$.
I.e., $F(x)=P\{X \leq x\}=$ ?
(2) (6 points) $E X^{4}=$ ?
(3) (6 points) Var $X=$ ?
(4) (6 points) Define $Y=\frac{1}{X+1}$. Give the moment generating function (MGF) of $Y$, $m_{Y}(t)=$ ?.
(5) (4 points)Find the $q$ th quantile of $X$ with $q=\frac{1}{4}$. (No work need be shown.)
II. (14 points.) $\left\{X_{1}, \ldots, X_{n}\right\}$ is an independent collection of random variables having finite mean $\mu$ and finite variance $\sigma^{2}$. Let $S=\sum_{k=1}^{3} k X_{k}$. Your answers to the questions below will involve $\mu$ and $\sigma$.
(1) (4 points.) $E(S)=$ ?
(2) (4 points) $\operatorname{Var}(S)=$ ?
(3) (6 points) If $Y=X_{1}+2 X_{2}-3$ and $Z=X_{1}-2 X_{2}+X_{3}+4$, then $\operatorname{Cov}(Y, Z)=$ ?
III. (21 points.) $X$ has probability density function (PDF)

$$
f_{X}(x)= \begin{cases}\frac{3 x^{2}}{8} & 0<x<2 \\ 0 & \text { otherwise }\end{cases}
$$

(1) (6 points) Give the distribution function (CDF) of $X$.
I.e., $F(x)=P\{X \leq x\}=$ ? (Be sure to give all pieces of this function. Otherwise you will lose points.)
(2) (6 points) $E X=$ ?
(3) (3 points) $P\{X=1 / 3\}=$ ?
(4) (6 points) $P\left\{X^{2}<\frac{1}{4}\right\}=$ ?
IV. (36 points.) $X$ has distribution function (CDF)

$$
F_{X}(x)= \begin{cases}0 & x<0 \\ \frac{1+x^{2}}{10} & 0 \leq x<2 \\ 1 & 2 \leq x\end{cases}
$$

(1) (5 points) Does this distribution function (CDF) have any jumps? If yes, identify all the jump points and the associated probabilities (i.e. $P(X=x)=$ ?).
(2) (5 points) $P\left\{-\pi<X \leq \frac{1}{2}\right\}=$ ?
(3) (5 points) $P\{0 \leq x \leq 2\}=$ ?
(4) (3 points) $A$ is the set $\left\{-1,0, \frac{1}{\sqrt{2}}, 2\right\} . P\{X \notin A\}=$ ?
(5) (2 points) Find $F^{\prime}(x)=$ ?
(6) (8 points) $E X=$ ?
(7) (8 points) $\operatorname{Var}(X)=$ ? (give a summation of reduced fractions as your final answer, no need to simplify.)
V. (30 points.) $X$ is normally distributed with mean $\mu=3$ and variance $\sigma^{2}=36$.
(1) (5 points) $P\{3 \leq X \leq 15\}=$ ?
(2) (5 points) $P\{|X-6| \leq 9\}=$ ?
(3) (5 points) Find out a value for $x_{0}$ so that $P\left\{X \geq x_{0}\right\}=0.99$
(4) (5 points) $P\left\{X^{3}+27 \geq 0\right\}=$ ?
(5) (5 points) Give the moment generating function (MGF) for $X, m_{X}(t)=$ ?
(6) (5 points) Let $Y=5 X+10$. Give the probability density function (PDF) for $Y$. I.e., $f_{Y}(x)=$ ?
VI. (28 points.) $X$ has a cumulative distribution function (CDF). (If you don't pay attention to the difference between CDF and PDF, you can say goodbye to the following 33 pts )

$$
F_{X}(x)= \begin{cases}1-e^{-5 x} & x>0 \\ 0 & x \leq 0\end{cases}
$$

(1) $(4$ points) $E(5 X+1)=$ ?
(2) $(4$ points $) \operatorname{Var}(5 X+1)=$ ?
(3) (5 points) Give the moment generating function (MGF) of $X, m_{X}(t)=$ ?
(4) (5 points) $E\left[(5 X)^{20}\right]=$ ?
(5) (5 points) $P\left\{\left.X \leq \frac{1}{10} \right\rvert\, X \geq \frac{1}{20}\right\}=$ ?
(6) (5 points) Let $Y=10 X$. Give the distribution function (CDF) of $Y$. I.e., $F_{Y}(y)=$ ? Moreover, based on $F_{Y}(y)$, tell what distribution $Y$ has (and what is/are the parameter(s)).
VII. (28 points.) $X$ and $Y$ have joint probability mass function given in the table below.

| $y \backslash^{x}$ | 1 | 2 | otherwise |
| :---: | :---: | :---: | :---: |
| 0 | $\frac{1}{10}$ | $\frac{2}{10}$ | 0 |
| 1 | $\frac{3}{10}$ | $\frac{4}{10}$ | 0 |
| otherwise | 0 | 0 | 0 |

(1) (5 points) $E(X Y)=$ ?
(2) (5 points) $\operatorname{Cov}(X, Y)=?$
(3) (5 points) Are $X$ and $Y$ independent of each other? (give your arguments. no argument, no point.)
(4) (8 points) Give $P\{Y=y \mid X=x\}$ for all values of $x$ and $y$. (You need to organize them as a "table".)
(5) (5 points) $E\{Y \mid X=1\}=$ ?
VIII. (33 points.) $X$ and $Y$ have joint probability density function (PDF)

$$
f(x, y)= \begin{cases}\frac{3}{4} y & 0<x<2,0<y<2, x+y<2 \\ 0 & \text { otherwise }\end{cases}
$$

(1) (5 points) Give the (marginal) probability density (marginal PDF) $f_{X}(x)$ of $X$. (Be sure to specify the regions of $x$.)
(2) (5 points) Give the (marginal) probability density (marginal PDF) $f_{Y}(y)$ of $Y$. (Be sure to specify the regions of $y$.)
(3) (4 points) Are $X$ and $Y$ independent of each other? (give your arguments. no argument, no points!)
(4) (5 points) $P(X>Y)=$ ? (Leave your answer as an integral. Make sure your limits of integration are correct.)
(5) (6 points) Give the conditional probability density function (conditional PDF) of $Y$ given that $X=x$. I.e., $f_{Y \mid X}(y \mid x)=$ ? (Be sure to specify the region of $(x, y)$ in which your result holds.)
(6) (4 points) $E\left\{Y \left\lvert\, X=\frac{1}{2}\right.\right\}=$ ?
(7) (4 points) $E\left\{(Y+1) X \left\lvert\, X=\frac{1}{2}\right.\right\}=$ ?

## Standard Normal Probabilities



Table entry for $z$ is the area under the standard normal curve to the left of $z$.

| $z$ | . 00 | . 01 | . 02 | . 03 | . 04 | . 05 | . 06 | . 07 | . 08 | . 09 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | . 5000 | . 5040 | . 5080 | . 5120 | . 5160 | . 5199 | . 5239 | . 5279 | . 5319 | . 5359 |
| 0.1 | . 5398 | . 5438 | . 5478 | . 5517 | . 5557 | . 5596 | . 5636 | . 5675 | . 5714 | . 5753 |
| 0.2 | . 5793 | . 5832 | . 5871 | . 5910 | . 5948 | . 5987 | . 6026 | . 6064 | . 6103 | . 6141 |
| 0.3 | . 6179 | . 6217 | . 6255 | . 6293 | . 6331 | . 6368 | . 6406 | . 6443 | . 6480 | . 6517 |
| 0.4 | . 6554 | . 6591 | . 6628 | . 6664 | . 6700 | . 6736 | . 6772 | . 6808 | . 6844 | . 6879 |
| 0.5 | . 6915 | . 6950 | . 6985 | . 7019 | . 7054 | . 7088 | . 7123 | . 7157 | . 7190 | . 7224 |
| 0.6 | . 7257 | . 7291 | . 7324 | . 7357 | . 7389 | . 7422 | . 7454 | . 7486 | . 7517 | . 7549 |
| 0.7 | . 7580 | . 7611 | . 7642 | . 7673 | . 7704 | . 7734 | . 7764 | . 7794 | . 7823 | . 7852 |
| 0.8 | . 7881 | . 7910 | . 7939 | . 7967 | . 7995 | . 8023 | . 8051 | . 8078 | . 8106 | . 8133 |
| 0.9 | . 8159 | . 8186 | . 8212 | . 8238 | . 8264 | . 8289 | . 8315 | . 8340 | . 8365 | . 8389 |
| 1.0 | . 8413 | . 8438 | . 8461 | . 8485 | . 8508 | . 8531 | . 8554 | . 8577 | . 8599 | . 8621 |
| 1.1 | . 8643 | . 8665 | . 8686 | . 8708 | . 8729 | . 8749 | . 8770 | . 8790 | . 8810 | . 8830 |
| 1.2 | . 8849 | . 8869 | . 8888 | . 8907 | . 8925 | . 8944 | . 8962 | . 8980 | . 8997 | . 9015 |
| 1.3 | . 9032 | . 9049 | . 9066 | . 9082 | . 9099 | . 9115 | . 9131 | . 9147 | . 9162 | . 9177 |
| 1.4 | . 9192 | . 9207 | . 9222 | . 9236 | . 9251 | . 9265 | . 9279 | . 9292 | . 9306 | . 9319 |
| 1.5 | . 9332 | . 9345 | . 9357 | . 9370 | . 9382 | . 9394 | . 9406 | . 9418 | . 9429 | . 9441 |
| 1.6 | . 9452 | . 9463 | . 9474 | . 9484 | . 9495 | . 9505 | . 9515 | . 9525 | . 9535 | . 9545 |
| 1.7 | . 9554 | . 9564 | . 9573 | . 9582 | . 9591 | . 9599 | . 9608 | . 9616 | . 9625 | . 9633 |
| 1.8 | . 9641 | . 9649 | . 9656 | . 9664 | . 9671 | . 9678 | . 9686 | . 9693 | . 9699 | . 9706 |
| 1.9 | . 9713 | . 9719 | . 9726 | . 9732 | . 9738 | . 9744 | . 9750 | . 9756 | . 9761 | . 9767 |
| 2.0 | . 9772 | . 9778 | . 9783 | . 9788 | . 9793 | . 9798 | . 9803 | . 9808 | . 9812 | . 9817 |
| 2.1 | . 9821 | . 9826 | . 9830 | . 9834 | . 9838 | . 9842 | . 9846 | . 9850 | . 9854 | . 9857 |
| 2.2 | . 9861 | . 9864 | . 9868 | . 9871 | . 9875 | . 9878 | . 9881 | . 9884 | . 9887 | . 9890 |
| 2.3 | . 9893 | . 9896 | . 9898 | . 9901 | . 9904 | . 9906 | . 9909 | . 9911 | . 9913 | . 9916 |
| 2.4 | . 9918 | . 9920 | . 9922 | . 9925 | . 9927 | . 9929 | . 9931 | . 9932 | . 9934 | . 9936 |
| 2.5 | . 9938 | . 9940 | . 9941 | . 9943 | . 9945 | . 9946 | . 9948 | . 9949 | . 9951 | . 9952 |
| 2.6 | . 9953 | . 9955 | . 9956 | . 9957 | . 9959 | . 9960 | . 9961 | . 9962 | . 9963 | . 9964 |
| 2.7 | . 9965 | . 9966 | . 9967 | . 9968 | . 9969 | . 9970 | . 9971 | . 9972 | . 9973 | . 9974 |
| 2.8 | . 9974 | . 9975 | . 9976 | . 9977 | . 9977 | . 9978 | . 9979 | . 9979 | . 9980 | . 9981 |
| 2.9 | . 9981 | . 9982 | . 9982 | . 9983 | . 9984 | . 9984 | . 9985 | . 9985 | . 9986 | . 9986 |
| 3.0 | . 9987 | . 9987 | . 9987 | . 9988 | . 9988 | . 9989 | . 9989 | . 9989 | . 9990 | . 9990 |
| 3.1 | . 9990 | . 9991 | . 9991 | . 9991 | . 9992 | . 9992 | . 9992 | . 9992 | . 9993 | . 9993 |
| 3.2 | . 9993 | . 9993 | . 9994 | . 9994 | . 9994 | . 9994 | . 9994 | . 9995 | . 9995 | . 9995 |
| 3.3 | . 9995 | . 9995 | . 9995 | . 9996 | . 9996 | . 9996 | . 9996 | . 9996 | . 9996 | . 9997 |
| 3.4 | . 9997 | . 9997 | . 9997 | . 9997 | . 9997 | . 9997 | . 9997 | . 9997 | . 9997 | . 9998 |

