A Bernstein-Type Inequality for Vector Functions on Finite Markov Chains

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Abstract

An analogue of the Bernstein inequality is derived for partial sums of a vector-valued function on a finite reversible Markov chain. The inequality gives an upper bound for the probability of a large deviation of the partial sum. The bound depends on the chain's spectral gap, the dimension of the space where the function takes values, and the upper bound on the size and the variance of the function.

1 Result

Let $\mathbb{S}$ be the state space of a finite Markov chain with transition matrix P and stationary distribution μ. Let the eigenvalues of P arranged in the declining order be

$$\lambda_1 = 1 > \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_{|S|} > -1.$$

Call the difference $g = 1 - \lambda_2$ the *spectral gap* of the chain. Recall that the chain is called reversible if $\mu_s P_{st} = \mu_t P_{ts}$ for any s and t from $\mathbb{S}$. In this paper we discuss only reversible chains.

Let f be a function on $\mathbb{S}$ that takes values in an m-dimensional real Hilbert space H. We use $\langle , \rangle$ and $|\cdot|$ to denote the scalar product and norm in this Hilbert space. We study the behavior of the norm of partial sums $S_N = \sum_{t=1}^{N} f(s_t)$.

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The behavior of S_N depends on the initial distribution $\mu^{(0)}$ and properties of the function f. Define the following distance:

$$\left\|\mu^{(0)}/\mu\right\| =: \left(\sum_s \left[\frac{\mu^{(0)}(s)}{\mu(s)}\right]^2 \mu(s)\right)^{1/2},$$

and the following measure of the volatility of f:

$$\sigma^2 = \sup_{|u|=1} \sum_s \mu_s \langle f(s), u\rangle^2,$$

which we call the *principal variance* of f.

Here is the main result. For every $\alpha \in (0,1)$ define the following *convergence rate*:

$$\kappa(\alpha) = \left[2\sigma^2\left(1+\frac{2}{g}\right) + \frac{4(1-\alpha)}{\alpha^3} g \log^{-2}\left[1+\frac{g}{2}\right]\right]^{-1}.$$

Theorem 1 *Suppose 1) P is reversible, 2) f has the zero mean relative to the stationary distribution: $\sum_s \mu_s f(s) = 0$, 2) f has the principal variance σ^2, and 4) f is uniformly bounded: $|f(s)| \leq L$. For arbitrary $\alpha \in (0,1)$ and $\varepsilon > 0$ define*

$$\overline{\varepsilon} = \min\left(\frac{\varepsilon}{L}, (1-\alpha)\left[\sigma^2 + \frac{4(1-\alpha)}{\alpha^3}\right]\right).$$

Then

$$\Pr\{|S_N| \geq \varepsilon N\} \leq C(m,N,\alpha)\exp\left[-\kappa(\alpha)\overline{\varepsilon}^2 N\right],$$

where

$$C(m,N,\alpha) = \frac{12\left\|\mu^{(0)}/\mu\right\|}{\Gamma\left(\frac{m}{2}+1\right)\left(1-e^{-4\kappa(\alpha)\overline{\varepsilon}^2 N}\right)} \exp\left[\frac{3\left(m^2+2m\right)}{16\kappa(\alpha)\overline{\varepsilon}^2 N}\right]\left(\kappa(\alpha)\overline{\varepsilon}^2 N\right)^{\frac{m+1}{2}}$$

If in addition $N \geq \left(m^2+2m+4\right)/\left(8\kappa(\alpha)\overline{\varepsilon}^2\right)$, then the inequality holds with

$$C(m,N,\alpha) = \frac{64\left\|\mu^{(0)}/\mu\right\|}{\Gamma\left(\frac{m}{2}+1\right)}\left(\kappa(\alpha)\overline{\varepsilon}^2 N\right)^{\frac{m+1}{2}}.$$

Remarks:

1. For a fixed m the probability of large deviations declines exponentially with the rate at least $-\kappa(\alpha)\overline{\varepsilon}^2$. Note that this bound on the rate does not depend on the dimension of the Hilbert space. However, the dimension can significantly affect the constant before the exponential.

2. Note that for a small g we have

$$\kappa(\alpha) \sim \frac{g}{4} \frac{1}{\sigma^2 + 4(1-\alpha)/\alpha^3}.$$

3. The bound depends on the free parameter α. It seems that a good rule of thumb is to take the largest possible $\alpha < 1$ such that it is still true that $(1-\alpha)\left[\sigma^2 + 4(1-\alpha)/\alpha^3\right] \geq \varepsilon/L$.

Example 2 *Random Walk on a Circle*

Suppose P describes a random walk on a circle that consists of n states. If the current state is $x \in \{1, ..., n\}$, then the next state is $x \pm 1 \bmod(n)$ with probability 1/2 on each possibility. Let n be odd so that P is aperiodic and irreducible. In particular take $n = 11$. The stationary distribution is uniform and the spectral gap $g = 1 - \cos(\pi/n) \approx 0.04$ [see Diaconis (1988) or Saloff-Coste (2004)]. Assume that the random walk starts from the uniform distribution. Take $L = 1$, $\sigma^2 = 1$, $\varepsilon = 0.01$, and $m = 1$. Choose $\alpha = 0.99$, then $\overline{\varepsilon} = 0.01$, and the convergence rate $\kappa = 9.5 \times 10^{-3}$. Then for all $N \geq 9 \times 10^6$ we have the following bound:

$$\Pr\{|S_N| \geq 0.01N\} \leq 0.05.$$

If the dimension of the Hilbert space is $m = 10$ then this bound holds for $N \geq 1.9 \times 10^7$.

Example 3 *Random Walk on a Hypercube*

Let the state space be the set of vertices of a k-dimensional hypercube. With probability $1/(k+1)$ the next state will be one of the k adjacent vertices, and with probability $1/(k+1)$ it remains the same. The spectral gap is $g = 1/(k+1)$ [see Diaconis (1988) or Saloff-Coste (2004)]. Let us consider dimension $k = 4$ so that the number of vertices in the cube is 16. Assume that the random walk starts from the uniform distribution. Take $L = 1$, $\sigma^2 = 1$, and $m = 1$. Then $g \approx 0.12$. For $\alpha = 0.99$ the convergence rate is $\kappa = 2.7 \times 10^{-2}$. Then the bound

$$\Pr\{|S_N| \geq 0.01N\} \leq 0.05$$

holds for all $N \geq 3 \times 10^6$. If $m = 10$ then it holds for $N \geq 7 \times 10^6$.

Explicit bounds on the probability of large deviations of vector-valued sums are needed in applications of the Monte Carlo Markov Chain method. In these applications, we aim to compute the average value of a vector-valued random variable. While it might be difficult to draw directly from the distribution of this random variable, it is often possible to construct a Markov chain that has this distribution as its stationary distribution. Then the question arises about the speed of convergence of averages of random draws from this chain to the expected value computed using the stationary distribution.

To put the problem in a more general context, let me sketch its history and relations with other questions in this area. Bernstein proved his inequality (for i.i.d. variables) in 1924 (see Paper 5 in Bernstein (1952)). Two well-known generalizations were derived by Chernoff (1952) and Hoeffding (1963) who relaxed the assumption of boundedness of the summands. Prokhorov (1968) proved a multi-dimensional analogue of the Bernstein inequality for i.i.d. random variables. An infinite-dimensional analogue was proved by Yurinskii (1970) . The inequality has also been generalized to the case of martingales.

Miller (1961) first studied the asymptotic behavior of additive real-valued functionals on finite Markov chains. Very definitive and general results in this direction were later obtained by Donsker and Varadhan (1975). Their results are valid for vector-valued or even measure-valued functionals of Markov chains acting on state spaces from a very general class. While results of this type are useful for understanding the asymptotic behavior of large deviations, they do not provide explicit bounds of deviations in finite samples.

The central limit theorem (CLT) for Markov chains goes back to Markov and was thoroughly explored by Nagaev (1957), among others. For a recent study of the speed of convergence in CLT see the dissertation by Mann (1996).

There is also a large body of recent literature about explicit rates of convergence of the initial distribution of a Markov chain to its stationary distribution. For a review, see the book by Diaconis (1988), the review paper by Saloff-Coste (2004), and the dissertation by Gangolli (1991).

The first one-dimensional Bernstein-type inequality for finite Markov chains was proved by Gillman (1993) (see also Dinwoodie (1995) and Lezaud (1998) for improvements). Gillman uses the method of generating functions and results from Kato's theory of perturbations of linear operators. (This method has also been used by Nagaev (1957) to study the CLT for Markov chains). This method of generating functions is not directly applicable to the multi-dimensional situation. To circumvent this difficulty we use a trick invented by Prokhorov (1968), which he used to prove the multi-dimensional analogue of the Bernstein inequality for i.i.d. random variables.

For dimension $m = 1$ we can compare our bound with some other bounds in the literature. Gillman (1993) shows that

$$\Pr\{S_N \geq \varepsilon N\} \leq 2 \left\| \mu^{(0)}/\mu \right\| \exp\left[-\frac{g}{20\nu} \varepsilon^2 N \right]$$

where ν is the spread of P, that is, $\nu = \max(\mu)/\min(\mu)$. Our bound from Theorem 1 does not depend on ν. In addition, even if $\nu = 1$ (the best case for Gillman's result), our bound is still better, provided ε is sufficiently small and α is taken sufficiently close to 1.

Bounds in Lezaud (1998), similarly to our bounds, depend on the variance of function f. For small ε and small g, Lezaud gives the following estimate of the convergence rate $\kappa \sim g/4\sigma^2$, which coincides with our rate. Dinwoodie (1995) estimates the convergence rate as $g/2$ for small ε, which is better than our rate if σ is large.

2 Proof

2.1 Outline

Let

$$B_r(x) = \int_{|u|\leq r} \exp\langle x, u\rangle\, du,$$

where x and u are vectors from H, and du is the Borel measure. Note that $B_r(x)$ depends only on $|x|$, that is, $B_r(x) = B_r(y)$ provided that $|x| = |y|$. Consequently, we can find a function of one variable $f(t)$ so that $B_r(x) = f(|t|)$. Somewhat abusing notation, we will also denote this function B_r. It is clear that $B_r(|x|)$ is an increasing positive function of $|x|$. Consequently we can write:

$$\begin{aligned} \Pr\{|S_N| \geq \varepsilon N\} &= \Pr\{B_r(|S_N|) \geq B_r(\varepsilon N)\} \\ &\leq \frac{1}{B_r(\varepsilon N)} \mathrm{E}\{B_r(|S_N|)\} \\ &= \frac{\int_{|u|\leq r} \mathrm{E}\exp\langle S_N, u\rangle\, du}{B_r(\varepsilon N)}, \end{aligned} \tag{1}$$

where the expectation E is relative to the initial distribution $\mu^{(0)}$.

The idea is to bound $B_r(\varepsilon N)$ from below by an exponential with the rate which is linear in r, then bound the integral from above by an exponential with the rate which is quadratic in r, and then find an optimal r that would minimize the ratio of these expressions.

2.2 Lower Bound on B_r

In this section we estimate the denominator of the ratio in (1). First, the denominator can be expressed as a Bessel function of imaginary argument.

Lemma 4

$$B_r(x) = \left[\frac{2\pi r}{|x|}\right]^{m/2} I_{m/2}(r|x|).$$

Remark: Here I_ν is a modified Bessel function. It is related to the usual Bessel function as follows:

$$I_\nu(z) = e^{-\nu\pi i/2} J_\nu(z e^{\pi i/2}) \text{ for } \arg z \in [-\pi; \pi/2],$$

and has the following asymptotic expansion

$$I_{m/2}(z) = \frac{e^z}{\sqrt{2\pi z}} \left(1 - \frac{m^2 - 1}{8z} + \frac{\left(m^2 - 1\right)\left(m^2 - 9\right)}{2!\,(8z)^2} - ... \right)$$

[see Watson (1948)].

Proof of Lemma: Using substitution $u = rv$, we obtain the following expression:

$$\begin{aligned} B_r(x) &= r^m \int_{|v| \le 1} \exp \langle rx, v \rangle \, dv \\ &= \left[\frac{2\pi r}{|x|} \right]^{m/2} I_{m/2}(r\,|x|), \end{aligned}$$

where the last equality uses Sonin's formula [see Fikhtengoltz (1949), p. 483]. □

Lemma 5 *Suppose ν is an integer or half-integer number and $\nu \ge 1/2$. Denote the integer part of ν as n. If $x > 0$ then*

$$I_\nu(x) \ge \frac{(e^x - e^{-x})}{\sqrt{2\pi x}} \exp \left[- \frac{3(n^2 + n)}{2x} \right]$$

If $x \ge n^2 + n + 1$ then

$$I_\nu(x) \ge \frac{1}{6\sqrt{2\pi x}} e^x.$$

Proof: A result by Cochran (1967) shows that if $\nu \ge 0$ then modified Bessel functions $I_\nu(x)$ decrease in their order ν (see also Olver (1997), Theorem 8.1 in Chapter 7). So

$$I_n(x) \ge I_{n+1/2}(x).$$

Also, it is known that $I_{1/2}(x) = \left(e^x - e^{-x}\right) / \sqrt{2\pi x}$. Consequently we need to prove only that for any integer n

$$I_{n+1/2}(x) \ge I_{1/2}(x) \exp \left(- \frac{3(n^2 + n)}{2x} \right).$$

We will prove this inequality using results by Nåsell (1974). In particular, we use inequality (11) in Nåsell (1974):

$$I_{\nu+1}(x) \ge \left[\frac{1 + 2(\nu + 1)/x + 2(\nu + 1)(\nu + 3/2)/x^2}{1 + (\nu + 3/2)/x} \right]^{-1} I_\nu(x).$$

The expression in square brackets can be re-written as follows:

$$\begin{aligned}\frac{1+(\nu+1)\frac{2}{x}+(\nu+1)(\nu+\frac{3}{2})\frac{2}{x^2}}{1+(\nu+\frac{3}{2})\frac{1}{x}} &= 1+\left(\nu+\frac{1}{2}\right)\frac{1}{x}\frac{1+\frac{(\nu+1)(\nu+3/2)}{\nu+1/2}\frac{2}{x}}{1+\left(\nu+\frac{3}{2}\right)\frac{1}{x}} \\ &\leq 1+3\left(\nu+\frac{1}{2}\right)\frac{1}{x},\end{aligned}$$

because

$$1+\frac{(\nu+1)\,(\nu+3/2)}{\nu+1/2}\frac{2}{x}\leq 3\left(1+(\nu+3/2)\,\frac{1}{x}\right),$$

provided that $\nu \geq 1/2$. Consequently, we have the following simplified bound:

$$I_{\nu+1}(x)\geq\left[1+3\left(\nu+\frac{1}{2}\right)\frac{1}{x}\right]^{-1}I_\nu(x)$$

for $\nu \geq 1/2$. Then

$$\begin{aligned}\frac{I_{n+1/2}}{I_{1/2}} &\geq \left[\prod_{k=1}^{n}\left(1+3\frac{k}{x}\right)\right]^{-1} \\ &= \exp\left[-\sum_{k=1}^{n}\log\left(1+3\frac{k}{x}\right)\right] \\ &\geq \exp\left[-\frac{3}{x}\sum_{k=1}^{n}k\right] \\ &= \exp\left[-\frac{3n(n+1)}{2x}\right],\end{aligned}$$

which is exactly what we need. The bound for $x \geq n^2+n+1$ is an elementary consequence of the general bound. □

Corollary 6

$$B_r(x)\geq\frac{(2\pi r)^{\frac{m-1}{2}}}{|x|^{\frac{m+1}{2}}}\exp\left[-\frac{3(m^2+2m)}{8r\,|x|}\right]\left(e^{r|x|}-e^{-r|x|}\right).$$

If $4r\,|x|\geq m^2+2m+4$ *then*

$$B_r(x)\geq\frac{(2\pi r)^{\frac{m-1}{2}}}{6\,|x|^{\frac{m+1}{2}}}e^{r|x|}.$$

2.3 Relation to eigenvalues of the perturbed transition matrix

Now turn to the estimation of the integral in the numerator of (1), that is, of

$$\int_{|u|\le r} \mathrm{E}\exp\langle S_N, u\rangle\, du.$$

We will first write the integrand as a quadratic form and then apply the Courant-Fischer theorem to reduce the problem to the estimation of the largest eigenvalue of this form.

Define the perturbed transition matrix as follows:

$$P_{st}(u) = \exp\langle f(t), u\rangle\, P_{st}.$$

Let us use $(,)$ to denote the following scalar product: $(a, b) = \sum_s a_s b_s$, where s denote the states of the chain and a_s and b_s are scalar-valued functions of s. Also let us denote the scalar-valued function that takes the value 1 on all states as $\mathbf{1}$.

Lemma 7

$$\mathrm{E}\exp\langle S_N, u\rangle = \left(\mu^{(0)}, [P(u)]^n\, \mathbf{1}\right)$$

Proof: We can write:

$$\begin{aligned}
\mathrm{E}\exp\langle S_N, u\rangle &= \sum_{s_0, s_1, \ldots, s_N} \mu^{(0)}_{s_0} P_{s_0 s_1} e^{\langle f(s_1), u\rangle} \ldots P_{s_{N-1} s_N} e^{\langle f(s_N), u\rangle} \\
&= \sum_{s_0, s_1, \ldots, s_N} \mu^{(0)}_{s_0} P_{s_0 s_1}(u) \ldots P_{s_{N-1} s_N}(u) \\
&= \left(\mu^{(0)}, [P(u)]^n\, \mathbf{1}\right).
\end{aligned}$$

□

Let us denote the largest eigenvalue of $P(u)$ as $\lambda(u)$. The Courant-Fischer theorem is not directly applicable because $P(u)$ is not symmetric. However, because of reversibility we are still able to bound the form in terms of the largest eigenvalue.

Lemma 8 *If the chain P is reversible, $|u| \le 1$ and $|f(s)| \le 1$ for any s, then*

$$\left(\mu^{(0)}, [P(u)]^n\, \mathbf{1}\right) \le 3 \left\|\mu^{(0)}/\mu\right\| \lambda(u)^n$$

Proof: Define $D = \operatorname{diag}\left\{\sqrt{\mu_s}\right\}$. Then reversibility of P implies that $S =: DPD^{-1}$ is symmetric. Indeed,

$$\begin{aligned} S_{ji} &\equiv \mu_j^{1/2} P_{ji} \mu_i^{-1/2} = \mu_j^{-1/2} \mu_j P_{ji} \mu_i^{-1/2} \\ &= \mu_j^{-1/2} \mu_i P_{ij} \mu_i^{-1/2} = S_{ij.} \end{aligned}$$

Also, define $E_u = \operatorname{diag}\left\{\exp \frac{1}{2} \langle f(s), u \rangle\right\}$ and $S_u = E_u S E_u$. Evidently, S_u is symmetric. Moreover it is conjugate to $P(u)$:

$$\begin{aligned} P(u) &\equiv PE_u^2 = D^{-1} S D E_u^2 \\ &= D^{-1} E_u^{-1} \left(E_u S E_u\right) E_u D \\ &= \left(E_u D\right)^{-1} S_u \left(E_u D\right), \end{aligned}$$

where we have used commutativity of D and E_u. Consequently S_u and $P(u)$ have the same eigenvalues. In particular, $\lambda(u)$ is the largest eigenvalue of S_u.

Next we re-write the form in question in terms of matrix S_u and apply the Courant-Fisher theorem to estimate it:

$$\begin{aligned} \left(\mu^{(0)}, [P(u)]^n \mathbf{1}\right) &= \left(\mu^{(0)} \left(E_u D\right)^{-1}, S_u^n \left(E_u D\right) \mathbf{1}\right) \\ &\leq \lambda(u)^n \left\|\mu^{(0)} \left(E_u D\right)^{-1}\right\| \left\|\left(E_u D\right) \mathbf{1}\right\| \\ &= \lambda(u)^n \left(\sum_s \frac{\left[\mu_s^{(0)}\right]^2}{\mu_s} \exp \langle -f(s), u \rangle\right)^{1/2} \\ &\quad \times \left(\sum_s \mu_s \exp \langle f(s), u \rangle\right)^{1/2} \\ &\leq 3 \left\|\mu^{(0)}/\mu\right\| \lambda(u)^n. \end{aligned}$$

In the inequality on the last line we have used the fact that $|\langle f(s), u \rangle| \leq |f(s)| \, |u| \leq 1$ and consequently $\exp \langle \pm f(s), u \rangle \leq 3$. □

2.4 A bound on the largest eigenvalue of the perturbed transition matrix

We need to estimate the largest eigenvalue of the perturbed transition matrix by a quadratic function of the size of the perturbation. The main idea is to estimate the second derivative of the eigenvalue as a function of the perturbation parameter. It is relatively easy to bound the second derivative at the zero perturbation by developing the eigenvalue function in a series in perturbation parameter (the Rellich method). It is somewhat more difficult to estimate the second derivative in a neighborhood near the zero perturbation. This is best done by studying properties of the resolvent of the perturbed operator in the complex plane (the Kato method, see Kato (1980)). We use this method to estimate the third derivative and then integrate the estimate using also the estimate of the second derivative at the zero perturbation (obtained using Rellich's method).

We have $P(\varkappa) = PV(\varkappa)$, where $V(\varkappa) = \operatorname{diag}\{\exp\langle f(s), \varkappa u\rangle\}$, and $|u| = 1$. We can write $P(\varkappa)$ as a power series in $\varkappa$:

$$P(\varkappa) = P + \varkappa PV^{(1)} + \varkappa^2 PV^{(2)} + ..., \tag{2}$$

where

$$V^{(n)} = \operatorname{diag}\left\{\frac{\langle f(j), u\rangle^n}{n!}\right\}.$$

For convenience, we call **Asssumption A** the following set of conditions:

1. P is a reversible chain with spectral gap g,
2. the expectation of f is zero: $\sum_s \mu_s f(s) = 0$,
3. the principal variance of f is σ^2: $\sup_{|u|=1} \sum_s \mu_s \langle f(s), u\rangle^2 = \sigma^2$,
4. $|f(s)| \leq 1$ for each s, and
5. the perturbed matrix $P(\varkappa)$ is given by (2) with $|u| = 1$.

We study the dependence of the largest eigenvalue of $P(\varkappa)$ on the perturbation parameter $\varkappa$. First, what is the first derivative of the largest eigenvalue at the zero perturbation?

Lemma 9 *Let Assumption A hold. Then*

$$\left.\frac{d\lambda(\varkappa)}{d\varkappa}\right|_{\varkappa=0} = 0.$$

Proof: Consider the power series expansions around $\varkappa = 0$. The expansion for the perturbed operator is (2), and the expansions for the largest eigenvalue and the corresponding eigenvector are

$$\begin{aligned} \lambda(\varkappa) &= 1 + \varkappa\lambda^{(1)} + \varkappa\lambda^{(2)} + ..., \\ X(\varkappa) &= \mu + \varkappa X^{(1)} + \varkappa X^{(2)} + ... \end{aligned}$$

Writing the equality $X(\varkappa)P(\varkappa) = \lambda(\varkappa)X(\varkappa)$ in powers of $\varkappa$, we get:

$$\begin{aligned} \mu P &= \mu, \\ X^{(1)}P + \mu P V^{(1)} &= \lambda^{(1)}\mu + X^{(1)}. \end{aligned} \tag{3}$$

Multiply the last line by $\mathbf{1}$ on the right and use the facts that $P\mathbf{1} = \mathbf{1}$ and $\mu\mathbf{1} = 1$. Then we get:

$$\lambda^{(1)} = \mu V^{(1)}\mathbf{1}.$$

However, $\mu V^{(1)}\mathbf{1} = \sum_s \mu_s \langle f(s), u\rangle = 0$ by assumption. Therefore, $\lambda^{(1)} = 0$. $\square$

We also need some information about the perturbation of the eigenvector, in particular, about $X^{(1)}$. From (3), it must satisfy the following equation:

$$X^{(1)}(I - P) = \mu V^{(1)}. \tag{4}$$

Note, however, that if vector X satisfies this equation then $X + a\mu$ also satisfies it. We define $X^{(1)}$ as the unique solution that satisfies the additional condition that $\left(X^{(1)}, \mathbf{1}\right) = 0$.

Let $\mu^\vee$ be the subspace of vectors orthogonal to $\mathbf{1}$. This subspace is invariant under the right action of P. Indeed, if $(x, 1) = 0$ then $(xP, 1) = (x, P1) = (x, 1) = 0$. Let us define operator R, which is an inverse of $I - P$ on $\mu^\vee$:

$$R = \sum_{k=0}^{\infty} \left(P|_{\mu^\vee}\right)^k.$$

Here $P|_{\mu^\vee}$ is the operator that multiplies vectors from $\mu^\vee$ by P on the right. The powers of this operator are well defined because of the invariance of $\mu^\vee$ relative to the right action by P, and the sum converges because all the eigenvalues of the right action by P are less than 1 on $\mu^\vee$ (by Perron-Frobenius theorem and because $\mu \notin \mu^\vee$).

If P reversible then $P = D^{-1}SD$, where S is symmetric. The subspace $\mu^\vee D^{-1}$ and its orthogonal complement are invariant under the action of S and we can define

$$T = \sum_{k=0}^{\infty} \left(S|_{\mu^\vee D^{-1}} \right)^k,$$

which acts on $\mu^\vee D^{-1}$. Let us extend T to the full space by declaring $T = 0$ on the orthogonal complement to $\mu^\vee D^{-1}$. Note that T and S commute: $TS = ST$. Furthermore, $D^{-1}TD$ is defined on $\mu^\vee$ and coincides there with R.

Lemma 10 *Let Assumption A hold. Then*

$$X^{(1)} = \mu V^{(1)} R = \mu V^{(1)} D^{-1} T D$$

Note that the product $\mu V^{(1)} R$ is well defined because $\left(\mu V^{(1)}, \mathbf{1}\right) = \sum_s \mu_s \langle f(s), u \rangle = 0$ and therefore $\mu V^{(1)} \in \mu^\vee$.

Proof: The product $\mu V^{(1)} R$ satisfies equation (4) and belongs to $\mu^\vee$. Consequently, it coincides with $X^{(1)}$. $\square$

Now consider the second derivative of the eigenvalue function:

Lemma 11 *Let Assumption A hold. Then*

$$\left| \left. \frac{d^2 \lambda(\varkappa)}{d\varkappa^2} \right|_{\varkappa=0} \right| \leq \left(1 + \frac{2}{g}\right) \sigma^2.$$

Proof: Let us write terms before $\varkappa$ and $\varkappa^2$ in the expansion of the equality $X(\varkappa)P(\varkappa) = \lambda(\varkappa)X(\varkappa)$, taking into account that $\lambda^{(1)} = 0$:

$$\begin{aligned} X^{(1)}P + \mu P V^{(1)} &= X^{(1)}, \\ X^{(2)}P + X^{(1)}PV^{(1)} + \mu P V^{(2)} &= \lambda^{(2)}\mu + X^{(2)}. \end{aligned}$$

Multiplying the second equality by $\mathbf{1}$ on the right we get the following formula for $\lambda^{(2)}$:

$$\lambda^{(2)} = \mu V^{(2)} \mathbf{1} + X^{(1)} P V^{(1)} \mathbf{1}. \tag{5}$$

Consider the absolute value of the second term in (5):

$$\begin{aligned} \left| X^{(1)} P V^{(1)} \mathbf{1} \right| &= \left| \mu V^{(1)} D^{-1} T S D V^{(1)} \mathbf{1} \right| \\ &\leq \|TS\| \left\| \mu V^{(1)} D^{-1} \right\| \left\| D V^{(1)} \mathbf{1} \right\|, \end{aligned}$$

where we used Lemma 10 and the equality $P = D^{-1}SD$. (Here we use $\|\cdot\|$ to denote both the norm of functions on the Markov chain and the induced norm of operators acting on these functions: by definition $\|f\| = (f, f)^{1/2}$ and $\|A\| = \sup_{\|f\|=1} \|Af\|$.

Operator TS is symmetric with eigenvalues which are either zeros or $\lambda_i / (1 - \lambda_i)$, where $i > 1$. Consequently

$$\|TS\| \leq \frac{1}{g}.$$

Next,

$$\left\| DV^{(1)}\mathbf{1} \right\| = \left(\sum_s \mu_s \langle f(s), u \rangle^2 \right)^{1/2} \leq \sigma,$$

and

$$\left\| \mu V^{(1)} D^{-1} \right\| = \left(\sum_s \mu_s \langle f(s), u \rangle^2 \right)^{1/2} \leq \sigma,$$

where we used that $D = \operatorname{diag}\left\{ \sqrt{\mu_s} \right\}$. In total,

$$\left| X^{(1)} P V^{(1)} \mathbf{1} \right| \leq \frac{\sigma^2}{g}.$$

Finally, for the first term in (5) we have

$$\left| \mu V^{(2)} \mathbf{1} \right| = \left| \frac{1}{2} \sum \mu_s \langle f(s), u \rangle^2 \right| \leq \frac{\sigma^2}{2},$$

and therefore

$$\lambda^{(2)} \leq \sigma^2 \left(\frac{1}{2} + \frac{1}{g} \right).$$

□

Now we turn to the estimation of the third derivative of the eigenvalue function in the neighborhood of zero. The following is a quick excursion in Kato's theory of perturbations. Let $T(\varkappa)$ be a perturbed operator:

$$T(\varkappa) = T + \varkappa T^{(1)} + \varkappa^2 T^{(2)} + ...,$$

where $\varkappa$ is the perturbation parameter. Let us for economy of space write $A(\varkappa) = \varkappa T^{(1)} + \varkappa^2 T^{(2)} + ...$

We want to find out for which $\varkappa$ the resolvent of the perturbed operator $R(\zeta, \varkappa) \equiv [T(\varkappa) - \zeta]^{-1}$ is non-singular provided that we know that the resolvent of unperturbed operator, $R(\zeta) \equiv [T - \zeta]^{-1}$, is non-singular at ζ.

We can write

$$\begin{aligned} T(\varkappa) - \zeta &= T - \zeta + A(\varkappa) \\ &= (T - \zeta)\left[1 + R(\zeta)A(\varkappa)\right], \end{aligned}$$

and consequently,

$$R(\zeta, \varkappa) = \left[1 + R(\zeta)A(\varkappa)\right]^{-1} R(\zeta).$$

The power series for $\left[1 + R(\zeta)A(\varkappa)\right]^{-1}$ are convergent provided that

$$\|R(\zeta)A(\varkappa)\|_{sp} < 1, \tag{6}$$

where $\|\cdot\|_{sp}$ denotes the *spectral norm*:

$$\|X\|_{sp} \equiv \lim \sup_{n\to\infty} \|X^n\|^{1/n}.$$

In our case, the perturbation is given by the following series

$$P(\varkappa) = P + \varkappa P V^{(1)} + \varkappa^2 P V^{(2)} + ...,$$

where

$$V^{(n)} = \operatorname{diag}\left\{\frac{\langle f(j), u\rangle^n}{n!}\right\}.$$

Recall that the reversibility of P implies that it can be represented as $P = D^{-1}SD$, where $D = \operatorname{diag}\left\{\sqrt{\mu_s}\right\}$ and S is symmetric. Let us denote $(S-\zeta)^{-1}$ by $R_S(\zeta)$.

Lemma 12 *Let Assumption A hold. Then the power series for* $\left[1 + R(\zeta)A(\varkappa)\right]^{-1}$ *converges if* $|\varkappa| < \log\left(1 + \|R_S(\zeta)S\|^{-1}\right)$.

Proof: By (6) and (2) we should check that $\left\|R(\zeta)P\sum_{k=1}^{\infty}\varkappa^k V^{(k)}\right\|_{sp} < 1$. For reversible P we can write

$$R(\zeta)P \equiv (P-\zeta)^{-1}P = D^{-1}(S-\zeta)^{-1}SD = D^{-1}R_S(\zeta)SD.$$

Using the fact that both D and the perturbation $\sum_{k=1}^{\infty}\varkappa^k V^{(k)}$ are diagonal and, therefore, commute, we can further write:

$$R(\zeta)P\sum_{k=1}^{\infty}\varkappa^k V^{(k)} = D^{-1}R_S(\zeta)S\sum_{k=1}^{\infty}\varkappa^k V^{(k)}D.$$

Next, we use the property of the spectral norm that it is not changed by similarity transformations and write:

$$\begin{aligned}\left\| R(\zeta)P\sum_{k=1}^{\infty}\varkappa^k V^{(k)}\right\|_{sp} &= \left\| R_S(\zeta)S\sum_{k=1}^{\infty}\varkappa^k V^{(k)}\right\|_{sp} \\ &\leq \left\| R_S(\zeta)S\sum_{k=1}^{\infty}\varkappa^k V^{(k)}\right\|,\end{aligned}$$

where we also used the fact that the spectral norm is bounded from above by the usual operator norm. We can continue as follows:

$$\left\| R_S(\zeta)S\sum_{k=1}^{\infty}\varkappa^k V^{(k)}\right\| \leq \left\| R_S(\zeta)S\right\| \sum_{k=1}^{\infty}|\varkappa|^k \left\| V^{(k)}\right\|.$$

From assumptions on u and $f(s)$ it follows that $\left\| V^{(k)}\right\| \leq 1/k!$, and consequently,

$$\left\| R_S(\zeta)S\right\| \sum_{k=1}^{\infty}|\varkappa|^k \left\| V^{(k)}\right\| \leq \left\| R_S(\zeta)S\right\| \left(e^{|\varkappa|}-1\right).$$

This expression is less than 1, provided that $|\varkappa| < \log\left(1+\left\| R_S(\zeta)S\right\|\right)$. □

Lemma 13 *Suppose Assumption A holds. Let* Γ *be a circle of radius* r_Γ *in* ζ*-plane that has exactly one eigenvalue of* P *inside it. Define*

$$r = \min_{\zeta\in\Gamma}\log\left(1+\left\| R_S(\zeta)S\right\|^{-1}\right)$$

Then for every $\varkappa$ *such that* $|\varkappa| \leq r$*, there is exactly one eigenvalue of* $P(\varkappa)$ *inside* Γ.

Moreover, for each $\alpha \in (0,1)$ *and those* $\varkappa$ *that* $|\varkappa| \leq (1-\alpha)r$*, the eigenvalue function* $\lambda(\varkappa)$ *is holomorphic and the following estimate holds for its third derivative:*

$$\left|\frac{d^3\lambda(\varkappa)}{d\varkappa^3}\right| \leq \frac{12}{\alpha^3}\frac{r_\Gamma}{r^3}.$$

Proof: Let D be a circle in $\varkappa$-plane with center at 0 and radius $r = \log\left(1+\left\| R_S(\zeta)S\right\|^{-1}\right)$. Consider an arbitrary $\varkappa_0$ inside D. We can connect $\varkappa = 0$ and $\varkappa_0$ by a curve Λ that lies completely inside the circle D. When we change $\varkappa$ along this curve, the eigenvalues of operator $P(\varkappa)$ follow paths that never intersect the circle Γ – we know this because by Lemma 12 the power series for the resolvent converge on

Γ for every $\varkappa \in \Lambda$. Consequently, the number of eigenvalues of operator $P(\varkappa_0)$ that are located inside Γ is conserved along the path Λ. It follows that $P(\varkappa_0)$ has exactly one eigenvalue inside Γ.

For the second part of the lemma, take an arbitrary $\varkappa_0$ such that $|\varkappa_0| \leq (1-\alpha)r$. Then exactly one eigenvalue of $P(\varkappa_0)$ is inside Γ. Consider the circle D_0 with center at $\varkappa_0$ and radius αr. This circle lies entirely inside the circle D and consequently for any $\varkappa \in D_0$ there is only one eigenvalue of $P(\varkappa)$ inside Γ. Hence,

$$|\lambda(\varkappa) - \lambda(\varkappa_0)| \leq 2r_\Gamma.$$

Recall that $\lambda(\varkappa)$ is holomorphic [see Kato (1980)] and estimate its third derivative at $\varkappa_0$ by using Cauchy's inequality:

$$\left| \left. \frac{d^3\lambda(\varkappa)}{d\varkappa^3} \right|_{\varkappa=\varkappa_0} \right| \leq 6 \frac{\max_{\varkappa \in D_0} |\lambda(\varkappa) - \lambda(\varkappa_0)|}{|\varkappa - \varkappa_0|^3} = 6 \frac{2r_\Gamma}{(\alpha r)^3} = \frac{12}{\alpha^3} \frac{r_\Gamma}{r^3}.$$

□

Lemma 14 *Let Assumption A hold. Let Γ be a circle of radius $r_\Gamma = g/2$ around the largest eigenvalue. Then*

$$\max_{\zeta \in \Gamma} \|R_S(\zeta) S\| \leq \frac{2}{g}.$$

Proof: Since S is similar to P, it has the same eigenvalues. Since S is symmetric, $R_S(\zeta)S$ is also symmetric and its norm coincides with the largest absolute value of its eigenvalues. Further, $R_S(\zeta)S$ has eigenvalues $(\lambda_i - \zeta)^{-1} \lambda_i$. It is easy to see that if $\zeta \in \Gamma$, then the maximum is reached for $i = 1$ and $\zeta = 1 - g/2$. A calculation gives:

$$\|R_S(\zeta)S\| \leq \frac{2}{g}.$$

□

Lemma 15 *Let Assumption A hold. Take $\alpha \in (0,1)$ and let $r = \log[1 + g/2]$. Then for any $\varkappa$ such that $|\varkappa| \leq (1-a)r$, the following inequalities hold:*

$$\left|\lambda'''(\varkappa)\right| \leq \frac{6}{\alpha^3} g \log^{-3}\left[1 + \frac{g}{2}\right].$$

Proof: Take $r_\Gamma = g/2$ in Lemma 13 and apply Lemma 14 to bound $\max_{\zeta\in\Gamma}\|R(\zeta)\|$.

□

Combining the previous lemmas, we get the following result.

Lemma 16 *Let Assumption A hold. Take* $\alpha \in (0,1)$ *Then for any* $\varkappa$ *such that* $|\varkappa| \le (1-\alpha)\log[1+g/2]$, *the following inequality holds:*

$$|\lambda(\varkappa)| \le 1 + \left[\sigma^2\left(\frac{1}{2}+\frac{1}{g}\right) + \frac{1-\alpha}{\alpha^3} g \log^{-2}\left[1+\frac{g}{2}\right]\right]\varkappa^2.$$

Proof: First, using Lemmas 11 and 15 we write:

$$\begin{aligned} \left|\lambda''(\varkappa)\right| &\le \sigma^2\left(1+\frac{2}{g}\right) + \int_0^\varkappa \lambda'''(t)dt \\ &= \sigma^2\left(1+\frac{2}{g}\right) + \frac{6}{\alpha^3} g \log^{-3}\left[1+\frac{g}{2}\right]\varkappa. \end{aligned}$$

And then, using Lemma 9 we get:

$$\begin{aligned} \left|\lambda'(\varkappa)\right| &\le \int_0^\varkappa \left|\lambda''(t)\right| dt \\ &\le \sigma^2\left(1+\frac{2}{g}\right)\varkappa + \frac{3}{\alpha^3} g \log^{-3}\left[1+\frac{g}{2}\right]\varkappa^2, \end{aligned}$$

and

$$\begin{aligned} |\lambda(\varkappa)| &\le 1 + \int_0^\varkappa \left|\lambda'(t)\right| dt \\ &\le 1 + \sigma^2\left(\frac{1}{2}+\frac{1}{g}\right)\varkappa^2 + \frac{1}{\alpha^3} g \log^{-3}\left[1+\frac{g}{2}\right]\varkappa^3. \end{aligned}$$

Using the condition $|\varkappa| \le (1-\alpha)\log[1+g/2]$, we further reduce this expression to:

$$|\lambda(\varkappa)| \le 1 + \left[\sigma^2\left(\frac{1}{2}+\frac{1}{g}\right) + \frac{1-\alpha}{\alpha^3} g \log^{-2}\left[1+\frac{g}{2}\right]\right]\varkappa^2.$$

□

Corollary 17 *Assume* P *is a reversible chain with spectral gap* g. *Let* $P(v) = P\text{diag}\{\exp\langle f(s), v\rangle\}$, *where* $|f(s)| \le 1$ *and the principal variation of* f *is* σ^2. *Take* $\alpha \in (0,1)$. *Then for every* v *such that* $|v| \le (1-\alpha)\log[1+g/2]$, *the following inequality holds for the largest eigenvalue of* $P(v)$*:*

$$\lambda(v) \le 1 + \left[\sigma^2\left(\frac{1}{2}+\frac{1}{g}\right) + \frac{1-\alpha}{\alpha^3} g \log^{-2}\left[1+\frac{g}{2}\right]\right]|v|^2.$$

Proof: The inequality follows if we take $\varkappa = |v|$ and $u = v/|v|$ in Lemma 16.

□

2.5 Bound on the integral and proof of the theorem

In this section we will complete the estimation of the integral in the numerator of (1). We will also complete the proof of the theorem by finding the optimal radius of the sphere over which we integrate. Let us use notations

$$\begin{aligned} k(\alpha) &\equiv \left[\sigma^2\left(\frac{1}{2}+\frac{1}{g}\right)+\frac{1-\alpha}{\alpha^3}g\log^{-2}\left[1+\frac{g}{2}\right]\right], \\ r_0(\alpha) &\equiv (1-\alpha)\log\left[1+\frac{g}{2}\right]. \end{aligned}$$

Note that $k(\alpha)$ is related to the convergence rate $\kappa(\alpha)$, which we defined in the Introduction, as follows: $\kappa(\alpha) = [4k(\alpha)]^{-1}$Also, let β_m denote the surface of the unit sphere in the m-dimensional real Hilbert space:

$$\beta_m = \frac{m\pi^{m/2}}{\Gamma\left(\frac{m}{2}+1\right)}.$$

Lemma 18 *Assume P is a reversible chain with spectral gap g. Let $P(v) = P\mathrm{diag}\{\exp\langle f(s), v\rangle\}$, where $|f(s)| \leq 1$ and the principal variation of f is σ^2, and let $\lambda(u)$ be its largest eigenvalue. Take $\alpha \in (0,1)$. If $r \leq r_0(\alpha)$ then*

$$\int_{|u|\leq r} \lambda(u)^N du \leq \frac{\beta_m r^m}{m}\exp\left[Nk(\alpha)r^2\right].$$

Proof: For simplicity we omit α in $k(\alpha)$ and $r_0(\alpha)$ in the following. Using Corollary 17 and changing to spherical coordinates we obtain:

$$\begin{aligned} \int_{|u|\leq r} \lambda(u)^N du &\leq \int_{|u|\leq r}\left[1+k|u|^2\right]^N du \\ &= r^m\int_{|v|\leq 1}\left[1+kr^2|v|^2\right]^N dv \\ &= \beta_m r^m\int_0^1\left[1+kr^2t^2\right]^N t^{m-1}dt. \end{aligned}$$

We apply inequality $1+x \leq \exp(x)$ valid for $a \geq 0$:

$$\begin{aligned} \beta_m r^m\int_0^1\left[1+kr^2t^2\right]^N t^{m-1}dt &\leq \beta_m r^m\int_0^1\exp\left[Nkr^2t^2\right]t^{m-1}dt \\ &\leq \frac{\beta_m r^m}{m}\exp\left[Nkr^2\right]. \end{aligned}$$

□

Let

$$
\begin{aligned}
\overline{\varepsilon} &\leq \min\left(\varepsilon, 2kr_0\right) \\
&= \min\left(\varepsilon, (1-\alpha)\left[\sigma^2\left(1+\frac{2}{g}\right)\log\left[1+\frac{g}{2}\right] + 2\frac{1-\alpha}{\alpha^3} g \log^{-1}\left[1+\frac{g}{2}\right]\right]\right).
\end{aligned}
$$

Lemma 19 *If P is reversible and f is uniformly bounded: $|f(s)| \leq 1$, with the principal variation of σ^2, then*

$$
\begin{aligned}
\Pr\left\{|S_N| \geq \varepsilon N\right\} &\leq \frac{6\left\|\mu^{(0)}/\mu\right\|}{\Gamma\left(\frac{m}{2}+1\right)2^m}\left(\frac{\overline{\varepsilon}^2 N}{k}\right)^{\frac{m+1}{2}} \\
&\quad \times \exp\left[\frac{3\left(m^2+2m\right)}{4}\frac{k}{\overline{\varepsilon}^2 N}\right]\left(1-e^{-N\overline{\varepsilon}^2/k}\right)^{-1}\exp\left[-\frac{\overline{\varepsilon}^2 N}{4k}\right].
\end{aligned}
$$

If in addition $N \geq k\left(m^2+2m+4\right)/(2\overline{\varepsilon}^2)$, then

$$
\Pr\left\{|S_N| \geq \varepsilon N\right\} \leq \frac{32\left\|\mu^{(0)}/\mu\right\|}{\Gamma\left(\frac{m}{2}+1\right)2^m}\left(\frac{\overline{\varepsilon}^2 N}{k}\right)^{\frac{m+1}{2}} \exp\left[-\frac{\overline{\varepsilon}^2 N}{4k}\right].
$$

Proof: Since $\overline{\varepsilon} \leq \varepsilon$ we have

$$
\Pr\left\{|S_N| \geq \varepsilon N\right\} \leq \Pr\left\{|S_N| \geq \overline{\varepsilon} N\right\}.
$$

We will bound the probability on the right-hand side of this inequality. By equation (1), Corollary 6, and Lemmas 7, 8, and 18, we can write:

$$
\begin{aligned}
\Pr\left\{|S_N| \geq \overline{\varepsilon} N\right\} &\leq \frac{1}{B_r(\varepsilon N)}\int_{|u|\leq r} \mathrm{E}\exp\left\langle S_N, u\right\rangle du \\
&\leq \frac{3\left\|\mu^{(0)}/\mu\right\|}{B_r(\overline{\varepsilon} N)}\int_{|u|\leq r}\lambda(u)^N du \\
&\leq \frac{3\left\|\mu^{(0)}/\mu\right\|\beta_m\left(2\pi\right)^{\frac{1-m}{2}}}{m} r^{\frac{m+1}{2}}\overline{\varepsilon}^{\frac{m+1}{2}}N^{\frac{m+1}{2}} \\
&\quad \times \exp\left[\frac{3\left(m^2+2m\right)}{8r\overline{\varepsilon}N}\right]\left(1-e^{-2N\overline{\varepsilon}r}\right)^{-1} \\
&\quad \times \exp\left(-N\overline{\varepsilon}r + Nkr^2\right),
\end{aligned}
$$

provided that $r \leq r_0$.

Set $r = \overline{\varepsilon}/(2k)$. This choice minimizes the rate in the last exponential making it equal $-\overline{\varepsilon}^2 N/(4k)$. The condition $r \leq r_0$ – needed for applicability of Lemma

18 – is satisfied by the definition of $\overline{\varepsilon}$. Then the inequality can be re-written as follows:

$$\begin{aligned}
\Pr\{|S_N| \geq \overline{\varepsilon}N\} &\leq \frac{6\pi \left\|\mu^{(0)}/\mu\right\| \beta_m}{m\,(4\pi k)^{\frac{m+1}{2}}} \overline{\varepsilon}^{m+1} N^{\frac{m+1}{2}} \\
&\quad \times \exp\left[\frac{3k\left(m^2+2m\right)}{4\overline{\varepsilon}^2 N}\right]\left(1-e^{-N\overline{\varepsilon}^2/k}\right)^{-1} \exp\left[-\frac{\overline{\varepsilon}^2}{4k}N\right] \\
&= \frac{3\sqrt{\pi}\left\|\mu^{(0)}/\mu\right\| \overline{\varepsilon}^{m+1} N^{\frac{m+1}{2}}}{\Gamma\left(\frac{m}{2}+1\right) 2^m k^{\frac{m+1}{2}}} \\
&\quad \times \exp\left[\frac{3k\left(m^2+2m\right)}{4\overline{\varepsilon}^2 N}\right]\left(1-e^{-N\overline{\varepsilon}^2/k}\right)^{-1} \exp\left[-\frac{\overline{\varepsilon}^2}{4k}N\right].
\end{aligned}$$

If the condition $4r\overline{\varepsilon}N \geq m^2+2m+4$ holds, that is, if $N \geq k\left(m^2+2m+4\right)/\left(2\overline{\varepsilon}^2\right)$, then this estimate simplifies (by an application of the second part of Corollary 6) to

$$\Pr\{|S_N| \geq \overline{\varepsilon}N\} \leq \frac{18\sqrt{\pi}\left\|\mu^{(0)}/\mu\right\| \overline{\varepsilon}^{m+1} N^{\frac{m+1}{2}}}{\Gamma\left(\frac{m}{2}+1\right) 2^m k^{\frac{m+1}{2}}} \exp\left[-\frac{\overline{\varepsilon}^2}{4k}N\right]$$

□

Now to prove the theorem we notice that for any f that satisfies condition $|f(s)| \leq L$ we can define $g = f/L$, for which $|g(s)| \leq 1$. Then

$$\Pr\{|S_N[f]| \geq \varepsilon N\} = \Pr\left\{|S_N[g]| \geq \frac{\varepsilon}{L}N\right\},$$

and we can apply Lemma 19 using ε/L instead of ε. Also note that elementary inequalities lead to the following estimate:

$$2kr_0 \geq (1-\alpha)\left[\sigma^2 + \frac{4(1-\alpha)}{\alpha^3}\right],$$

and therefore

$$\overline{\varepsilon} = \min\left(\frac{\varepsilon}{L}, (1-\alpha)\left[\sigma^2 + \frac{4(1-\alpha)}{\alpha^3}\right]\right)$$

also satisfies conditions of Lemma 19.

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