RESEARCH STATEMENT

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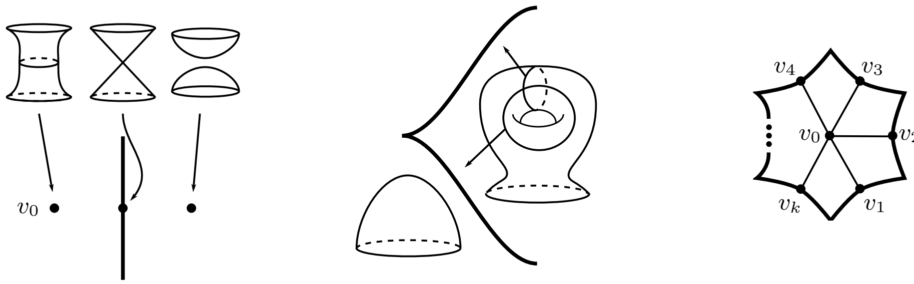
1. Background

Briefly, I have been working on perhaps the most famously difficult problem in 4-manifold topology: Find new tools, which researchers can actually use, to investigate the classification problem for smooth 4-manifolds. I have achieved this, and I am currently writing papers to present examples. This line of research is essentially unique, because most of the people in my field are unwilling to risk working seriously on this problem. This research statement includes lots of pictures to emphasize the hands-on, concrete character of what I am trying to do.

Where is the mystery in smooth 4-manifold classification? It has been known since the 1990s [GS] that there are smooth 4-manifolds M_1, M_2 such that M_1 and M_2 are equivalent as topological manifolds (that is, there is a homeomorphism $f\colon M_1 \to M_2$), but they are not *diffeomorphic*, that is, they are not equivalent as smooth manifolds (another way to say this is that f and f^{-1} cannot both be differentiable). For example, in every dimension besides $n = 4$, if a smooth manifold is homeomorphic to $\mathbb{R}^n$, then it is diffeomorphic to $\mathbb{R}^n$; however there are uncountably many pairwise non-diffeomorphic 4-manifolds homeomorphic to $\mathbb{R}^n$, so-called *exotic* $\mathbb{R}^4$*s*. Typically, M_1 is a well-known "standard" manifold coming from algebraic geometry, like $\mathbb{CP}^2$ or some other complex surface, and M_2 is an "exotic M_1" which is constructed using a variety of tools. To prove M_2 is exotic, one must associate some quantity to the diffeomorphism type of M_1 (that is, a smooth invariant) and prove the corresponding invariant of M_2 is different. Every method for finding exotic closed 4-manifolds thus far has revolved around the fact that the only readily computable invariant for smooth 4-manifolds is the Seiberg-Witten invariant which, though defined for arbitrary closed, smooth oriented 4-manifolds, is readily shown to be nontrivial only for the relatively small subset which admit symplectic forms. A symplectic form is a closed differential 2-form that is not the zero form at any point of the manifold. The Seiberg-Witten invariant of a smooth 4-manifold M is the collection of solutions to a nonlinear system of partial differential equations on M, and, roughly speaking, a symplectic form represents one solution. This fact has led to the construction of exotics, but the Seiberg-Witten invariant is known to vanish for large classes of 4-manifolds such as homology spheres and connected sums $M_1 \# M_2$, where each of M_1 and M_2 is a smooth 4-manifold with nonvanishing Seiberg-Witten invariant. Famously, the Seiberg-Witten invariant is not a useful tool for figuring out a very basic question: Is there an exotic 4-sphere? My research is aimed at finding a new, more generally nontrivial invariant for which calculations are possible. It is a program in two directions.



a. The image of a fold arc.

b. The image of a cusp point connecting two fold arcs.

c. The critical image of a crown map, decorated with a graph T.

FIGURE 1. The critical image of an indefinite Morse 2-function. Arrows and circles in depicted reference fibers indicate the attaching circles corresponding to indicated fold arcs.

2. SUMMARY OF ACCEPTED AND SUBMITTED PUBLICATIONS

2.1. **Morse 2-functions.** The first direction is to find better ways to depict any smooth, closed oriented 4-manifold M using *crown diagrams* and *crown multisection diagrams* coming from *indefinite Morse 2-functions* from M to the 2-sphere S^2 and to $\mathbb{R}^2$, respectively. Such a map $f\colon M \to S^2$ has a critical set comprising fold arcs and isolated cusp points, whose union is a smooth 1-submanifold of M, and whose image is depicted in Figure 1. The critical points of f (call the critical set $\operatorname{crit} f$) map to the bold arcs, and the pictured surfaces are mapped to the dots by f. As a dot gets closer to the bold critical arc, a 0-sphere or a 1-sphere in the fiber collapses to a single point, and the topology of the fiber changes as the dot crosses the bold arc. The distinguished circle in $f^{-1}(v_0)$ associated to an arc from v_0 to an arc of critical points is called a vanishing cycle.

In general, $\operatorname{crit} f$ may not be connected, and its image can have crossings. Also, $f(\operatorname{crit} f)$ can move around according to the Reidemeister moves when f is modified by a homotopy, and the associated vanishing cycles can interact in complicated ways. My paper [W1] proves that if a pair of such maps are homotopic, then the homotopy can be realized by a sequence chosen from a set of four local model homotopies. As an application, it also proves that any such map is homotopic to what I call a *crown map*, which is a map whose critical set is mapped to the 2-sphere in a pattern depicted in Figure 1c. The "spokes" of the wheel shown there form a graph T of reference paths which is not part of the critical image. For a crown map $f\colon M \to S^2$, the regular fiber $\Sigma = f^{-1}(v_0)$ can be decorated by the vanishing cycles γ_i obtained from using the depicted reference paths in T from v_0 to v_i, $i \in \{1, \dots, k\}$. The sequence of circles $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ is ordered cyclically. If Σ is at least 3, then f (and thus M) can be recovered from the pair (Σ, Γ). In that case, (Σ, Γ) is called a crown diagram.

My paper [W2] gives a uniqueness theorem for crown diagrams: There are four moves, or ways to modify a crown diagram without changing the 4-manifold it represents, and if (Σ, Γ) and (Σ', Γ') come from homotopic maps $M \to S^2$, then there is a sequence of moves that converts (Σ, Γ) into (Σ', Γ'). This paper originally appeared in 2011, and over the years (most recently April 2022) I have made extensive revisions in response to referee comments, significantly adding to its length and detail. Such a theorem is crucial for proving that some quantity associated to a crown diagram is actually a smooth 4-manifold invariant. The paper is an immensely technical and delicate argument, modifying the critical surface of a homotopy of maps $M \to S^2$ in a multi-stage process which, along the way, contains independent proofs of results that others over the years have decided to publish as papers in their own right; for example the modification of [W2, Lemma 3.8] appears as the main theorem in [BS].

As a reference for [W2] and for people who want to do constructions, my paper [W3] gives a reasonably universal library of situations in which a Reidemeister move can be achieved by a homotopy of f. As applications it builds on the work of [W2], refining the list of moves in various ways. For example, if the crown diagrams in the previous paragraph have the same genus, then the most mysterious and difficult of these moves on crown diagrams, stabilization, can be removed from the sequence of moves converting (Σ, Γ) to (Σ', Γ'). Stabilizations are difficult because, as currently described using equations in the mapping class groups of Σ and Σ', they involve adding an element to Γ which is defined in terms of all the other elements of Γ, and the calculation involved in this definition can be rather difficult. This result about omitting stabilizations for like-genus diagrams is a surprise because the analogous result for a close relative of crown diagrams (Heegaard diagrams of 3-manifolds) is false. There is another result stating that the remaining moves are sequences of what I call *slides*, as shown in Figure 2. These last two sentences are especially

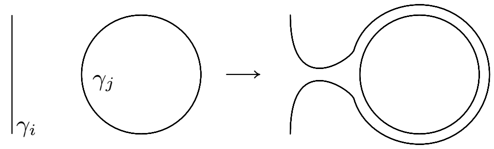


Figure 2. The vertical arc at left, which is part of a vanishing cycle γ_i, slides over the vanishing cycle γ_j at right. If a pair of diagrams differ by a sequence of slides, they are called *slide-equivalent*.

important for the second direction of research outlined below. Another application in that paper, depicted in [W2, Figure 7], is an existence result for crown diagrams in the form a basic, explicit algorithm to convert any continuous map $M \to S^2$ into a crown map.

The main result of [W2] involves moves one may perform on crown diagrams. These moves were closely studied in papers of Behrens and Hayano such as [BH], where they were given descriptions in the form of equations in mapping class groups of surfaces. Such descriptions are generally annoying to verify for a given modification of diagrams, and these equations lacked descriptions that actually showed how they change a given crown diagram. In [W2], I gave diagrammatic descriptions of the two less complicated moves, handleslide and multislide, and in [W4], I gave a diagrammatic description of the shift move. That paper also has a general diagrammatic framework for performing a move from [W1] called the cusp merge, which has historically mysterious for researchers who want to work with Morse 2-functions from the perspective of base diagrams such as Figure 1. The main motivation for this paper was to present an algorithm for converting a Lefschetz fibration over S^2 into a crown map, whose diagram I call a *coronation* of the Lefschetz fibration, while keeping track of all of the vanishing cycles, and to present a vastly simplified version of the algorithm that results in a diagram which is slide-equivalent to the coronation.

2.2. **Smooth invariants.** In [W5], I present the *salient set* of a crown diagram $(\Sigma, \Gamma = \{\gamma_1, \ldots, \gamma_k\})$. Its definition is simple enough to be given in the next paragraph. The important aspects of the salient set of a crown diagram are as follows:

- It is entirely combinatorial, avoiding the moduli spaces and perturbation theory of Seiberg-Witten and holomorphic curve techniques.
- It seems to be entirely applicable, because it does not require the geometric hypotheses of previous invariants. For this reason it seems realistically suited to addressing problems which have seemed entirely intractible for decades, such as the smooth 4-dimensional Poincare conjecture, which posits that there is no exotic 4-sphere.
- Its definition and calculation is elementary, allowing early graduate students and relatively simple software to work with it.
- It has elaborations as described in Section 3.

With that propaganda stated, here is the construction.

Because of the cyclic ordering on Γ, a crown diagram naturally specifies a k-component link in $\Sigma \times S^1$ given by the circles γ_i, and each crossing x in this link corresponds to a pair of complementary chords $\{x\} \times I$ and $\{x\} \times J$ in $\Sigma \times S^1$, where I, J are the two closed intervals from one strand of the crossing to the other. A choice of which circle is labeled γ_1 pins down which of the two chords appears as a vertical arc connecting the strands of each crossing. An example appears in Figure 3:

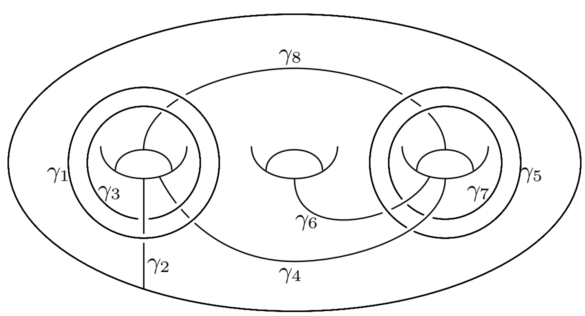


FIGURE 3. A crown diagram for the 4-sphere due to Hayano, with under- and over-crossings resulting from choosing a "first" vanishing cycle γ_1. The chords are line segments connecting the two strands of each crossing.

Each crossing now corresponds to exactly one chord, and the four quadrants near a crossing get signs as in Figure 4.

The link cuts Σ into regions with corners at the crossings, and each quadrant can be labeled by its corresponding signed chord. For each region, there is a *grading equation* obtained by setting the sum of the labels of its quadrants equal to zero.

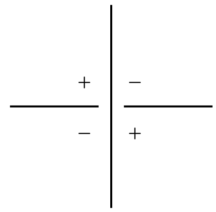


FIGURE 4. The signs of the quadrants near a crossing.

The system of equations obtained as the grading equations of all regions and for all choices of "first" vanishing cycles is called the *grading system* of (Σ, Γ). Observe that γ_i intersects γ_{i+1} at a unique transverse point, as required by the local model of a cusp point. There is a distinguished "short" chord traveling from γ_i to γ_{i+1}, without passing through any of the fibers $\Sigma \times \{pt\}$ containing the other components of the link; call these the *salient chords*. Then the salient set is the solution space of the grading system spanned by the salient chords.

In [W5], the slide invariance of the salient set is established with elementary linear algebra, and this implies that if $f_0, f_1 \colon M \to S^2$ are homotopic crown maps, then the salient sets of their crown diagrams are equal. Using software developed in collaboration with a Binghamton University computer science graduate student named Umur Ciftci, I was able to verify that the salient sets for crown diagrams of a pair of 4-manifolds which are known to have the same Seiberg-Witten invariants are not the same. For this reason, there is no smooth homotopy connecting their crown maps. If there were, then the corresponding smooth structures would be *isotopic* in the sense of smoothing theory, so the conclusion is that the smoothings represented by these manifolds are not isotopic. The isotopy class of a smooth structure is a much more subtle object than the diffeomorphism class, and it seems likely that the result can be strengthened to say the manifolds are not diffeomorphic; see Section 3.

3. Future

3.1. **Slide invariance conjecture.** As shown in [W6], there is a simple way to convert a crown map into a map to the disk while keeping track of the vanishing cycles. The critical image of the resulting map is a collection of concentric circles, with a circle c appearing like Figure 1c at the center. If $(\gamma_1, \gamma_2 \ldots, \gamma_k)$ was the sequence of vanishing cycles for the crown diagram, the sequence for c is $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k)$. There it is shown that the sequences coming from different choices of γ_1 are slide-equivalent. Given such a map, one may apply a homotopy as in Figure 5:

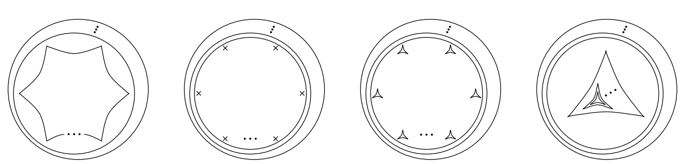


FIGURE 5. From left to right, converting a crown multisection map into a trisection map using well-understood moves.

This can all be done while keeping track of the vanishing cycles, according to the techniques of [W4]. Up to slide equivalence, each vanishing cycle becomes a triple as in Figure 6. In keeping the previous results of [W3, W4] concerning the slide equivalence of diagrams with the same genus, [W6] has the conjecture that any like-genus pair of crown diagrams that can be connected by the modifications of Figure 5 and the uniqueness of trisections given by [GK, Section 5] is slide equivalent. Such a result would allow the main result of [W5] to be strengthened to say the manifolds in that paper are not diffeomorphic, and it would allow the construction of salient sets to be applied in the setting of trisections and multisections of smooth 4-manifolds, which is a field with many active researchers who are writing papers with existence results, but essentially no uniqueness results due to the lack of useful invariants.

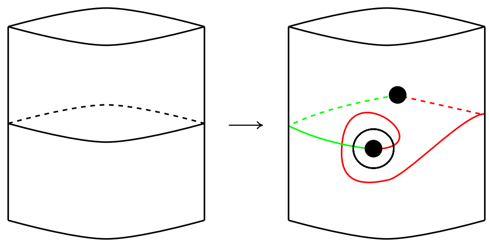


FIGURE 6. Conversion of crown diagram vanishing cycles into the vanishing cycles of a trisection.

3.2. **Differential graded algebra.** The chords of a crown diagram can be taken as the generators of an associative algebra over $\mathbb{Z}_2$, and this algebra has a differential defined exactly like the Legendrian contact homology differential from [C]. In that paper, Chekanov uses the condition that the components of a link are *Legendrian* to prove his differential squares to zero; in place of that, and in a rather straightforward way, one may instead use the fact that the components of the link specified by a crown diagram are *stacked*, in the sense that each lives in its own slice $\Sigma \times \{pt\} \subset S^1$. Legendrian contact homology has been an incredibly fruitful area of research for knot theory and contact topology, and the same can be expected for the differential chord algebra of a crown diagram.

3.3. **Topology of Morse 2-functions.** I plan to continue studying maps from 4-manifolds to surfaces. For example, I have recently found a new way to achieve the mysterious stabilization move mentioned at the beginning of this statement that allows one to use the techniques of [W4] to keep track of the vanishing cycles, as shown in Figure 7.

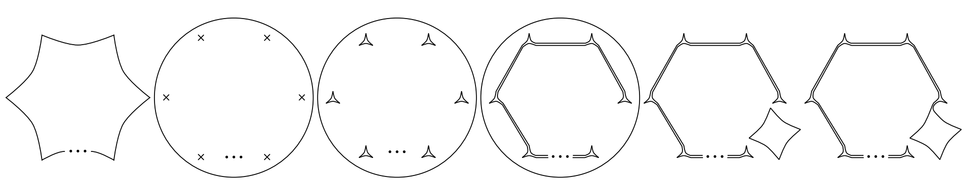


FIGURE 7. Stabilizing a crown map.

The moves in that figure are either well-understood with respect to vanishing cycles, or (as when circles merge into one) have been recently investigated by the author in [W4], so a simple diagrammatic description of the stabilization move for crown diagrams is now within reach.

Results such as this enable a true ability to work with maps from 4-manifolds to surfaces while keeping track of vanishing cycles. They will also be important for constructions and for comparing the salient sets and chord algebras for crown diagrams of different genus.

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