THE SALIENT CROSSINGS OF A CROWN MULTISECTION

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Abstract. A crown multisection of a smooth, closed oriented 4-manifold is a variation on Gay and Kirby's idea of a trisection. It comes from converting a crown map into a map to the disk in a certain controlled way, leading to a *crown multisection diagram* that records the relevant vanishing cycles. This sequence has a salient set (as in the author's previous work [W3]) which yields a smooth 4-manifold invariant capable of distinguishing the diffeomorphism classes of a pair of smooth 4-manifolds with the same Seiberg-Witten invariant.

Preliminary draft.

1. Introduction

1.1. **Motivation and overall goal of the paper.** Since the mid-1990s, the main tool for distinguishing pairs of smooth closed oriented 4-manifolds which are homeomorphic but not diffeomorphic has been the Seiberg-Witten invariant [GS, M]. Though calculations are generally considered to be reasonable, it offers no information in many interesting cases, for example homology spheres and connect sums of manifolds with positive b^+. This paper presents a restricted class of Morse 2-functions to the disk, called *crown multisection maps* (*crown* as in [W2], *multisection* as in [IN]), and a relatively elementary smooth 4-manifold invariant called the salient set $\mathcal{S}$ of a diagram associated to a crown multisection map. Using these, it becomes possible to prove the following theorem, assuming Conjecture 2.7 later in the paper:

Theorem 1.1. The Fintushel-Stern knot surgery 4-manifolds $E(1)_{7_6}$ and $E(1)_{10_{133}}$ are not diffeomorphic.

The significance of this theorem mainly comes from the fact that $E(1)_{7_6}$ and $E(1)_{10_{133}}$ have the same Seiberg-Witten invariant. This follows from [FS, Theorem 1.9], because the knots 7_6 and 10_{133} have the same Alexander polynomial [LM].

Finding a salient set is surprisingly straightforward: It is essentially achieved by row-reducing a large sparse matrix with entries in $\{-1, 0, 1\}$. The applicability of salient sets also seems quite broad: It is defined for any smooth closed oriented connected 4-manifold, and at present this author has no evidence that the salient set of a homology sphere, or any other smooth 4-manifold, should be trivial in any reasonable sense.

This paper is essentially a sequel, or possibly an appendix, to [W3], so that words such as *slide equivalence* and *salient set* are used as defined in that paper.

Acknowledgement. The author would like to thank Bob Gompf for pointing out that the salient set of a crown diagram [W3] might possibly only be capable of distinguishing isotopy classes of smooth structures, not diffeomorphism classes. This observation led the author to the present work.

2. Topological background

2.1. Crown multisection maps. This section assumes some familiarity with the theory of Morse 2-functions and trisections, well-developed in papers such as [BH, GK1, GK2, L]. To explain what a crown multisection map is, it seems most efficient to give its construction, which starts with a crown map:

Definition 2.1. Let M be a smooth, closed connected oriented 4-manifold. A *crown map* is a Morse 2-function $c: M \to S^2$ whose critical set is a single indefinite circle on which c is injective. Choosing a reference point at the center of the higher-genus region and using radial reference paths to the critical circle ordered counter-clockwise, the vanishing cycles appear as a cyclically ordered sequence $\Gamma = (\gamma_i)$, where $i \in \mathbb{Z}/k\mathbb{Z}$ and k is the number of cusps in the critical circle.

A crown multisection map is a restricted kind of radially monotonic Morse 2-function as in [IN]:

Definition 2.2. A *crown multisection map* is one whose base diagram appears as in Figure 6: There is one cusped central circle with no crossings bounding the highest-genus region, whose fibers are connected, and this circle is contained in a collection of concentric fold circles with no cusps and no crossings. Moving radially outward, the genus of the fiber decreases monotonically to 0, then shrinks to a point at the outermost circle, which is definite. Finally, the cusped circle has a sequence of vanishing cycles (γ_i) such that $\gamma_i = \gamma_{i+2}$ and $\gamma_{i+1} = \gamma_{i+3}$ for some i, and the vanishing cycles all lie in a genus $g+1$ subsurface of the reference fiber.

Definition 2.3. A genus-g *crown multisection diagram* is a pair (Σ, Γ), where Σ is the genus-$g > 2$ fiber above a point p in the highest-genus region of a crown multisection map and Γ is the cyclically ordered list of vanishing cycles of the innermost cusped circle, as measured using reference paths from p. A *stabilized* crown multisection diagram is a crown multisection diagram which has undergone some finite number of connect sums with copies of a manifold diffeomorphic to the pair

$$S = (\Sigma_1 = S^1 \times S^1, \{S^1 \times \{pt\}, \{pt\} \times S^1\}).$$

Taking the connect sum with S in the diagram corresponds to performing a birth move as in [L, Figure 12] centered at a point on the reference fiber. The notion of a stabilized crown multisection diagram will become relevant in Section 2.2.

Remark 2.4. A genus-g crown multisection diagram specifies a crown multisection map over a neighborhood of the highest-genus region. The rest of the map is a standard kind of fibration $H_{g-1} \times S^1 \to [0,1] \times S^1$, where H_{g-1} is a 3-dimensional genus $g-1$ handlebody. The gluing of these two pieces (and thus the rest of the map) is uniquely determined because the diffeomorphism group of Σ_{g-1} is simply-connected; for this reason, a crown multisection diagram specifies M up to diffeomorphism.

There is a simple way to convert a crown diagram into a crown multisection diagram:

Proposition 2.5. Suppose (Σ_g, Γ) is a crown diagram for M, where $\Gamma = (\gamma_1, \ldots, \gamma_k)$. Then (Σ_{2g+1}, Γ') is a crown multisection diagram for M, where

$$\Gamma' = (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_k).$$

Proof. The argument is given in Figures 2–6, in which a crown map is converted into a crown multisection map, while keeping track of the vanishing cycles. □

Remark 2.6. In Proposition 2.5, observe that the choice of which vanishing cycle in Γ should be labeled γ_1 was arbitrary, because the pair of cusps involved in the cusp merge was arbitrarily chosen. In other words, to convert a crown diagram into a crown multisection diagram, one may choose to repeat any pair of consecutive vanishing cycles. Conversely, for any crown multisection map, the sequence of moves from Figures 2–6 may be performed in reverse to obtain a projected crown map, corresponding to a crown diagram without the repeated pair of vanishing cycles.

2.2. **Uniqueness of crown multisection diagrams.** A sufficiently precise uniqueness statement for crown multisection diagrams is easily adapted from [IN, Theorem 8.5]. To paraphrase that result, any crown multisection map may be converted into a balanced trisection map by applying an un-sinking move as in [L, reverse of Figure 8] to each cusp as in [IN, Figure 25], then wrinkling the resulting Lefschetz critical points as in [L, Figure 12] and arranging the resulting critical triangles to be concentric (this is a sequence of what are called *UPW moves* in [IN]). With this understood, given two crown multisection maps $M \to D^2$, convert each to be a balanced trisection map. Then these maps are related by the uniqueness statement for trisections due to Gay and Kirby [GK2, Section 5]. Gay and Kirby's uniqueness involves two moves: slides between circles coming from the same sector, and a stabilization move coming from performing three birth moves [L, Figure 12] at the central highest-genus region. Each birth changes the diagram by connect sum with a torus decorated with a pair of simple closed curves with a single transverse intersection. This homotopy of converting to a balanced trisection map, performing slides and stabilizations, then converting back to a crown multisection map will be called an *equivalence* of crown multisection maps.

The following conjecture is work in progress:

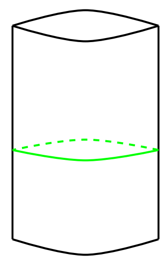
Conjecture 2.7. For any equivalence that does not involve the stabilization move, the resulting modification of crown multisection diagrams is realizable as a sequence of slides between vanishing cycles of the cusped circle of the original crown multisection map.

One may incorporate trisection stabilization into this regime as follows. Suppose a pair of crown multisection diagrams differ by an equivalence involving a stabilization. Because the birth move is supported in an arbitrarily small ball over the highest-genus region which is bounded away from all other critical points, one may perform the three birth moves of the stabilization before converting to a balanced trisection diagram: As with trisections, each birth changes the diagram by connect sum with a torus decorated with a pair of simple closed curves with a single transverse intersection. Next, perform slides within the trisection, then convert back to a crown multisection map with three birth moves applied, with a diagram which is slide-equivalent to the original thrice-birthed diagram. A bit more generally, there is no particular reason to perform three births: It makes just as much sense to perform a single birth in this paragraph. With the two previous paragraphs understood, we conclude:

Proposition 2.8. If two stabilized crown multisection diagrams of M have the same genus, then they are slide-equivalent.

3. The invariant

Definition 3.1. The salient set of a crown multisection diagram is the salient set of its associated crown diagram, as specified in Proposition 2.5. The salient set of a stabilized crown multisection diagram is that of the associated non-stabilized diagram, with a $\mathbb{Z}$-summand added for each stabilization.



Theorem 3.2. If two stabilized crown multisection diagrams of M have the same genus, then their salient sets are equal.

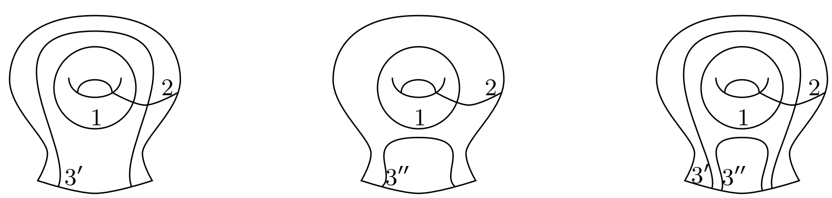
Proof. In converting a crown multisection diagram into a crown diagram, there may be more than one choice of a sequence of vanishing cycles which could be labeled $\gamma_1, \gamma_2, \gamma_1, \gamma_2$, so the first step is to prove this choice does not affect the resulting salient sequence. In this paragraph of the argument, for ease of reading, we use the notation i or i' instead of γ_i or γ_i', respectively. One approach is to observe that the sequence

$$1, 2, 1, 2, 3, 4, 5 \ldots, k$$

is slide-equivalent to the sequence

$$1, 2, 3, 2, 3, 4, 5, \ldots, k.$$

To prove this, start with a neighborhood of $1 \cup 2$ in Σ. Since $2 \cap 3$ is a single transverse point, after sliding 3 over 2 some number of times, the result is $3'$ as depicted in Figure 1a. Perform slides as in Figures 1b and 1c, and in the resulting



a. Slide 3 over 2 as needed to obtain $1, 2, 1, 2, 3'$.

b. Slide $3'$ over 1 to obtain $1, 2, 1, 2, 3''$.

c. Slide 1 over $3''$ to obtain $1, 2, 3', 2, 3''$.

Figure 1. These are figures for the first paragraph of the proof of Theorem 3.2, showing that crown diagrams which correspond to crown multisection diagrams in different ways are slide-equivalent.

diagram, both $3'$ and $3''$ can be converted to be 3 by sliding them over 1 and 2 as needed to obtain $1, 2, 3, 2, 3$. Repeating this process some finite number of times results in a diagram with a sequence $i, i+1, i, i+1, i, i+1$. At this point, it clearly does not matter which instance of $i, i+1$ or $i+1, i$ is removed. For this reason, the

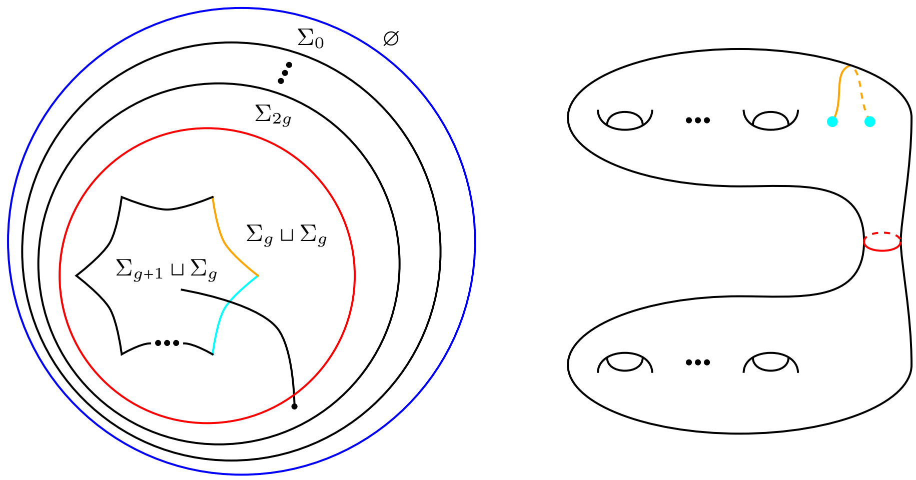


FIGURE 2. Applying the projection move from [W1, Section 4.1.2] to a crown map results in a *projected* crown map (base diagram at left, corresponding surgered diagram on the right). The colors of the vanishing cycles match those of their corresponding fold arcs. The vanishing cycles coming from the central cusped circle are all contained in the upper half of the reference fiber on the right; only two of them are pictured because the others remain unchanged by the following modifications. The outermost (blue) circle is definite, and moving out of that circle, the fibers change from $S^2 = \Sigma_0$ to $\varnothing$.

salient set of a crown diagram might as well be associated to any crown multisection diagram obtained by adding a repetition as in Proposition 2.5.

Thus, the different crown diagrams corresponding to a given stabilized crown multisection diagram are slide-equivalent. Since by Proposition 2.8 the various stabilized crown multisection diagrams of M with some fixed genus are also slide-equivalent, and salient sets are invariant under slide-equivalence, the theorem is proved. □

proof of Theorem 1.1. The salient sets of genus-5 crown diagrams for the manifolds in the theorem are shown to be different in [W3, Section 4.3], and so the salient sets of their corresponding genus-11 crown multisection diagrams are different. For this reason, these manifolds are not diffeomorphic. □

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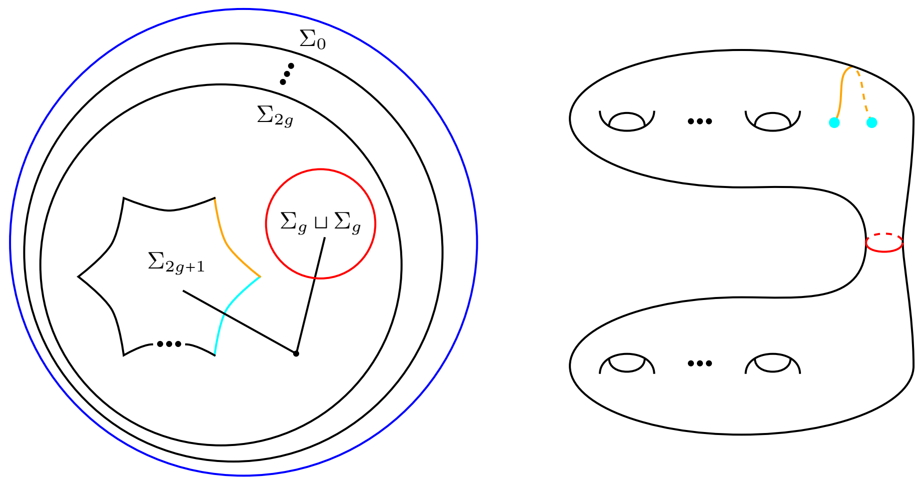

FIGURE 3. By [W2, Proposition 2.11], the cusped circle can be moved out of the innermost (red) circle that contains it in Figure 2.

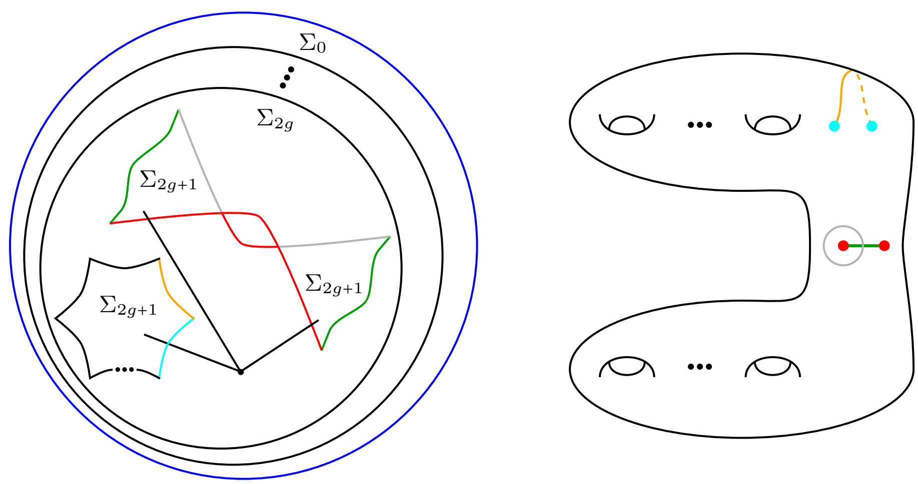

FIGURE 4. Now apply two flipping moves as in [L, Figure 12] to the smallest un-cusped (red) circle in Figure 3. The points inside the small bigon still have $\Sigma_g \sqcup \Sigma_g$ fibers, and as a reference point passes through either side of that bigon, the two fiber components undergo connect sum. The (red, green and grey) vanishing cycles for the two critical triangles are adapted to a surgered diagram presentation from their local model as depicted in [L, Figure 5].

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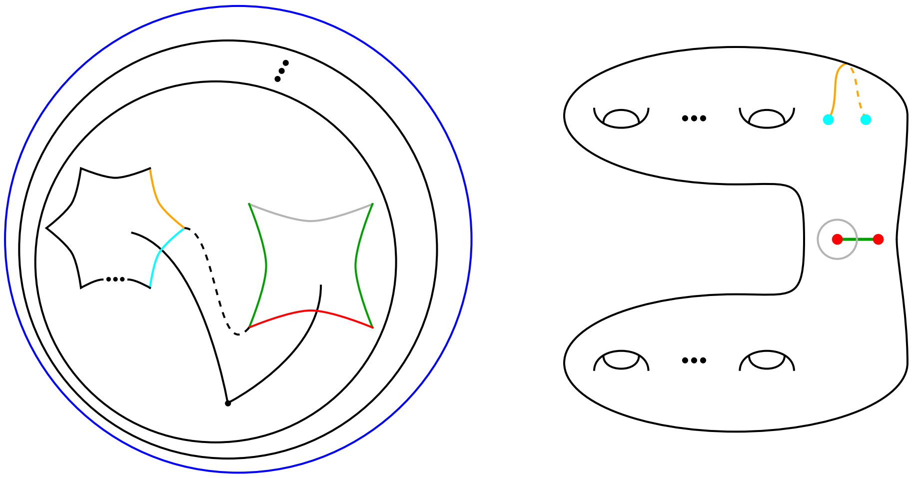

FIGURE 5. This figure results from an application of [W2, Proposition 2.5(3)] to the small bigon in Figure 4. The dotted line connecting the two cusps indicates an impending cusp merge, which exists by the arguments in the last paragraph of page 289 of [L]. In that paper, a cusp merge is called an *inverse merge*.

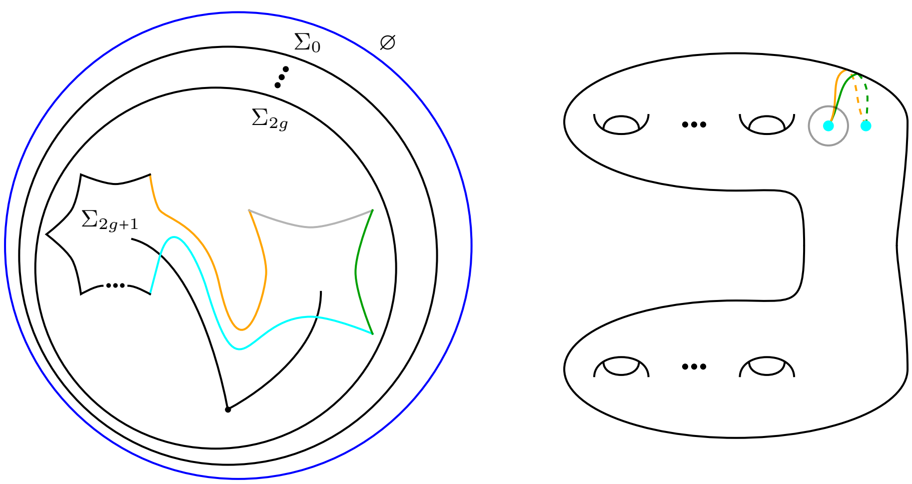

FIGURE 6. The cusp merge is completed. The vanishing cycles are as shown according to the technique described in [W4, Section 2.3]. Choosing a reference fiber in the highest-genus region, observe that the two new (grey and green) fold arcs introduced to the central circle are such that the vanishing cycle sequence $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ of the original crown diagram has become $\Gamma' = (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_k)$. In the surgered diagram at right, γ_1 appears both as the (light blue) pair of dots and the small (grey) circle around one of the dots.

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