Research Statement

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Overview: My research deals with a variety of topics in group theory. The research areas that I am currently active in include finite p-groups, cyclic word subgroups, and the Chermak-Delgado lattice. Broadly, I am interested in the structure of finite p-groups, the coclass conjectures, and investigating groups computationally with the aid of MAGMA and GAP. My PhD thesis work, supervised by Alexandre Turull, was concerned with classifying families of groups in the class of finite p-groups possessing an abelian subgroup of index p. A more recent project of mine, inspired by the work of I.D. Macdonald and in collaboration with Luise-Charlotte Kappe, dealt with investigating the number of n-th power elements necessary for generating a cyclic n-th power subgroup. An ongoing project, inspired by Ben Brewster and in collaboration with Lijian An, Elizabeth Wilcox, and Haipeng Qu, involved investigating the Chermak-Delgado lattice of finite p-groups.

1 Finite p-groups with an abelian subgroup of index p

Many of the existing determinations of prime-power groups take advantage of the existence of a subgroup whose index is a prime and the fact that all such subgroups are normal. The existence of such a subgroup allows usto view a p-group P of order p^n as a simple extension of a p-group Q of order p^{n-1} where $P = \langle Q, \alpha \rangle$ where $\alpha \in P - Q$. As abelian p-groups are the simplest non-cyclic p-group, a natural p-group to study is one for which Q is abelian; denote the class of such groups by $\mathcal{A}_p$.

In 1950 Hsio-Fu Tuan showed that if $P \in \mathcal{A}_p$ and A is abelian and maximal in P, then $A/Z(P) \cong P'$. Further, if $Z_i(P)$ and $L_i(P)$ denote terms of the upper and lower central series of P, respectively, then $A/(A \cap Z_i(P)) \cong L_{i+1}(P)$. As a maximal subgroup A of a p-group P is normal, conjugation by $x \in P - A$ will induce an automorphism on A, denoted σ_x, of order p. Denoting by 1_A and 0_A the identity and zero endomorphisms of A, respectively, σ_x satisfies the identity $\sigma_x^p - 1_A = 0_A$. Defining the A-endomorphisms

$$\phi_x = \sigma_x - 1_A \qquad\qquad \pi_x = 1_A + \sigma_x + \ldots + \sigma_x^{p-1}$$

G. Szekeres showed that the groups G which are non-abelian and have a normal abelian subgroup A with G/A cyclic such that if p is a prime divisor of $|A|$ then $p^2 \not| \, |G/A|$; this includes the class $\mathcal{A}_p$. Fixing $x \in P - A$, the A-endomorphisms ϕ_x and π_x are shown to be nilpotent maps and a $\mathbb{Z}[\phi_x, \pi_x]$-module is completely decomposable into indecomposable submodules C which are identified uniquely by being open or closed and are generated by a special basis called a *chain basis.*

Szekeres' characterization of $\mathbb{Z}[\phi_x, \pi_x]$-modules can be used to classify groups $P \in \mathcal{A}_p$ with an abelian maximal subgroup A, nilpotence class k, and rank m (that is $|P : \Phi(P)| = p^m$). p-Groups can have either $p+1$ or 1 abelian maximal subgroup and those with $p+1$ have been fully classified, hence we assume that A is unique.

As A is maximal we have $P = \langle x, A \rangle$ for any $x \in P - A$ and it can be shown that $x^p \in Z(P)$. With a collection of chain bases for A and a suitable $x \in P - A$ it is easy to obtain a power-commutator presentation for P. The difficulty that remains in the classification of $\mathcal{A}_p$ is determining which presentations are isomorphic or nonisomorphic. We now focus our attention on how a collection of chain bases can be found. The upper central series can be used to find collections of chain bases. The rank and abelian invariants of A can then be computed.

Future Plans: I'm currently in a position to answer questions concerning the class $\mathcal{A}_p$ posed by Berkovich: Classify the groups of $\mathcal{A}_p$ which are powerful p-groups, satisfy $|G'| = p^2$, or are of maximal class. Furthermore, I plan on studying the p-groups all of whose proper subgroups contain an abelian maximal subgroup.

2 Cyclic word subgroups

I.D. Macdonald asked the question if a cyclic commutator subgroup is always generated by a commutator. For every positive integer n, he provides examples of groups where the generators of the cyclic commutator subgroup are a product of not less than n commutators. This raises the following question for an n-variable word $f(x_1, \dots, x_n)$, the associated word subgroup $f(G)$ in a group G, and $F(G)$, the set of values of $f(x_1, \dots, x_n)$ in G:

If $f(G)$ is cyclic, under what conditions is $f(G)$ generated by an element in $F(G)$?

Luise-Charlotte Kappe and I gave an answer to the question for the 1-variable word $f(x) = x^n$, where n is a positive integer.

Example 2.1 (Brennan/Kappe). *Let $q \mid p-1$ for primes p, q and let $\alpha, \beta \geq 2$. Choose an integer γ where $\gamma - 1 \nmid p^\alpha$ and $\gamma^q - 1 \mid p^\alpha$. For $n \in \{p^\tau q^\sigma \mid 1 \leq \tau < \alpha, 1 \leq \sigma < \beta\}$ the group*

$$G = \left\langle a, b \mid a^{p^\alpha} = b^{q^\beta} = 1, a^b = a^\gamma \right\rangle$$

is such that G^n is cyclic, but G^n can be generated by two n-th powers of elements of G but cannot be generated by a single n-th power.

The questions arises if there exist groups with a cyclic G^n requiring a product of more than two elements from $G^{(n)}$ for its generators.

Theorem 2.2 (Brennan/Kappe). *If G^n is a finite cyclic group, then there exists $a, b \in G$ such that $G^n = \langle a^n, b^n \rangle$. If G^n is an infinite cyclic group, then there exists $g \in G$ such that $G^n = \langle g^n \rangle$.*

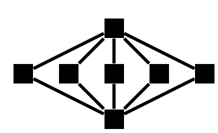
Future Plans: One might ask the question to which other words Macdonald's investigations could be extended. What comes to mind are 2-variable words where the associated varieties are well understood, such as the 2-Engel word $[x, y, y]$ and the 3-abelian word $x^3y^3(xy)^{-3}$, or groups where G'' is cyclic, i.e. the word $f(G) = G''$ with $f(x, y, z, u) = [[x, y], [z, u]]$. It is perhaps easy to conjecture that in all those cases there exist groups with cyclic $f(G)$ not generated by a single element in $F(G)$. If the number of factors from $F(G)$ is bounded or not awaits further investigations.

3 Chermak-Delgado lattice of finite groups

Let G be a finite group and $H \leq G$. The *Chermak-Delgado measure of H (in G)* is $m_G(H) = |H||C_G(H)|$. The $\mathcal{CD}$-measure was originally defined by A. Chermak and A. Delgado. The maximum $\mathcal{CD}$-measure possible in G is denoted by $m^*(G)$ and we use $\mathcal{CD}(G)$ to denote the set of all subgroups H with $m(H) = m^*(G)$. I. Martin Isaacs coined the term $\mathcal{CD}$-measure, and proved that subgroups with maximal $\mathcal{CD}$-measure form a modular sublattice in the subgroup lattice of the group. Ben Brewster and Elizabeth Wilcox then demonstrated that, for a direct product, this Chermak-Delgado lattice decomposes as the direct product of the $\mathcal{CD}$-lattices of the factors, giving rise to the attention on the Chermak-Delgado lattice of p-groups.

3.1 Chermake-Delgado Lattice Extension Theorems

The variety seen in the structure of the Chermak-Delgado lattices (a modular self-dual lattice) of p-groups seems inexhaustible. To describe the lattices constructed in this section, we introduce the following terms for a positive integer n. A *quasiantichain of width* n will be denoted by $\mathcal{M}_n$. This is formed by an antichain of n elements joined at a unique maximum element and a unique minimum element. An *n-diamond* is a lattice with subgroups in the configuration of an n-dimensional cube. A *n-string* is a lattice with n lattice isomorphic components, adjoined end-to-end so that the maximum of one component is identified with the minimum of the other component.



$\mathcal{M}_7$

Theorem 3.1 (Brennan/An/Wilcox/Qu). *Let N be a p-group such that $N \in \mathcal{CD}(N)$ and $\Phi(N) \leq Z(N)$. For any integers $n \geq 2$ and $m \geq 1$*

i) there exists a p-group P and a normal embedding of N into P, where $\mathcal{CD}(P)$ is a mixed 3-string with center component isomorphic to $\mathcal{CD}(N)$ and the remaining components being m-diamonds.

ii) there exists a p-group Q and a normal embedding of N into P, where $\mathcal{CD}(P)$ is a mixed 3-string with center component isomorphic to $\mathcal{CD}(N)$ and the remaining component $\mathcal{M}_{p+3}$.

In both of these groups, if two subgroups H and K with $H < K$ are in the Chermak-Delgado lattice and there does not exist a subgroup M in the Chermak-Delgado lattice such that $H < M < K$, then $|K : H| = p^n$.

3.2 Chermak-Delgado Simple p-Groups

Much of the previous work on the Chermak-Delgado measure has focused on determining which lattices arise as Chermak-Delgado lattices of finite groups. My work on extension theorems above implies that the shape of the $\mathcal{CD}$-lattice for a p-group of nilpotence class 2 can vary wildly. It is well known that the p-groups of nilpotence class 2 are a rich class of groups, so I've turned my attention to a class of p-groups with high nilpotence class and a restricted $\mathcal{CD}$-lattice.

A group G is *Chermak-Delgado Simple (or $\mathcal{CD}$-simple)* if $\mathcal{CD}(G) = \{G, Z(G)\}$.

Nonabelian $\mathcal{CD}$-simple groups do not possess large abelian subgroups. If P and $Z(P)$ are the only normal subgroups in $\mathcal{CD}(P)$, then P is $\mathcal{CD}$-simple. The following construction is a 2-group of order 2^{3n} with nilpotence class $2n-1$, $n \geq 2$, which is $\mathcal{CD}$-simple. $\mathcal{S}_p$ is also $\mathcal{CD}$-simple.

Construction 3.2. *Let $n \geq 2$ and let F be the free group on the generators $\{a, b, c, d, e\}$. Define the relations*

$$R_n = \{a^2, b^4, c^{2^n}, d^{2^{n-1}}, e^{2^{n-2}}, [b,a]c^{-1}, [c,b]d^{-1}, [c,a]c^2, [c,d]c^4e^{-2}, [d,b]ed^2, [c,e], [d,e][c,d]^2\}$$

and define the group $G(n) = F/\langle R_n \rangle^F$.

Definition 3.3. *Let $\mathcal{S}_p$ be the class of 2-generator p-groups P for which there exists $n \geq 1$ with*
(i) $|P : Z(P)| = p^{2n+1}$ (ii) nilpotence class $2n+1$ (iii) $C_P(Z_i(P)) = Z_{2n+2-i}(P)$ for any $1 \leq i \leq 2n+1$.

The lower p-exponent central series is defined $\lambda_0(P) = P$ and $\lambda_i(P) = \lambda_{i-1}(P)^p[\lambda_{i-1}(P), P]$ for $i \geq 1$. Assume that d is the smallest integer such that $\lambda_d(P) \leq Z(P)$, then $\lambda_d(P)$ and $Z(P)$ are not cyclic and if P has rank f then

$$\frac{\lambda_{d-1}(P)}{\lambda_{d-1}(P) \cap Z(P)}$$

is elementary abelian of rank r where $1 \leq r < f$.

Theorem 3.4. *Assume that $P \in \mathcal{S}_p$ with $n \geq 3$ is such that $\Phi(P) = P'$. Then the grandparent, $P/\lambda_{2n-1}(P)$, is an element of $\mathcal{S}_p$.*

Future Plans: The unpredictability of centralizers factored through quotients of groups presents a major difficulty to the study of the $\mathcal{CD}$-lattice. I intend to study the invariants of the $\mathcal{CD}$-lattice which are preserved through quotients of the lower exponent-p central series. MAGMA data seems to suggest that the below conjecture is true.

Conjecture 3.5. *A non-abelian $\mathcal{CD}$-simple p-group P which is a 2-generator group and has p-class c is such that $P/\lambda_{c-2}(P)$ is $\mathcal{CD}$-simple.*

In my next investigation of $\mathcal{S}_p$, I will be calculating their irreducible characters and studying their normal subgroup lattice. From MAGMA evidence, the number of normal subgroups of the groups $G(n)$ is $2n^2 - n + 11$, that is $|G(30)| = 2^{60}$ has 791 normal subgroups. The groups of $\mathcal{S}_p$ are "middle" nilpotence class p-groups, which typically have much larger than quadratic growth in their normal subgroups.