Research Statement

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Overview: My research deals with a variety of topics in group theory. The research areas that I am currently active in include finite p-groups, cyclic word subgroups, and the Chermak-Delgado lattice. Broadly, I am interested in the structure of finite p-groups, the coclass conjectures, and investigating groups computationally with the aid of MAGMA and GAP.

My PhD thesis work, supervised by Alexandre Turull, was concerned with classifying families of groups in the class of finite p-groups possessing an abelian subgroup of index p. A more recent project of mine, inspired by the work of I.D. Macdonald and in collaboration with Luise-Charlotte Kappe, dealt with investigating the number of n-th power elements necessary for generating a cyclic n-th power subgroup. An ongoing project, inspired by Ben Brewster and in collaboration with Lijian An, Elizabeth Wilcox, and Haipeng Qu, involved investigating the Chermak-Delgado lattice of finite p-groups.

1 Finite p-groups with an abelian subgroup of index p

Many of the existing determinations of prime-power groups take advantage of the existence of a subgroup whose index is a prime and the fact that all such subgroups are normal. The existence of such a subgroup allows usto view a p-group P of order p^n as a simple extension of a p-group Q of order p^{n-1} where $P = \langle Q, \alpha \rangle$ where $\alpha \in P - Q$. As abelian p-groups are the simplest non-cyclic p-group, a natural p-group to study is one for which Q is abelian; denote the class of such groups by $\mathcal{A}_p$.

The elements of $\mathcal{A}_p$ whose abelian maximal subgroups are cyclic were classified up to isomorphism by William Burnside in 1897. When $p = 2$ such groups consists of the generalized quaternion groups, 2-powered dihedral groups, and semi-dihedral groups. The earliest serious investigation of p-groups with an abelian maximal subgroup can be attributed to Phillip Hall [1] in 1940. Hall classified such groups up to isoclinism and made note that the sections of the lower central series were elementary abelian. Following this, in 1950 Hsio-Fu Tuan [2] expanded upon the structure of the upper and lower central series of groups in $\mathcal{A}_p$.

Theorem 1.1. *If $P \in \mathcal{A}_p$ and A is abelian and maximal in P, then $A/Z(P) \cong P'$. Further, if $Z_i(P)$ and $L_i(P)$ denote terms of the upper and lower central series of P, respectively, then*

$$A/(A \cap Z_i(P)) \cong L_{i+1}(P)$$

As a maximal subgroup A of a p-group P is normal, conjugation by $x \in P - A$ will induce an automorphism on A, denoted σ_x, of order p. Denoting by 1_A and 0_A the identity and zero endomorphisms of A, respectively, σ_x satisfies the identity $\sigma_x^p - 1_A = 0_A$. Defining the A-endomorphisms

$$\phi_x = \sigma_x - 1_A \qquad \pi_x = 1_A + \sigma_x + \ldots + \sigma_x^{p-1}$$

the identity can be factored

$$\sigma_x^p - 1_A = \phi_x \pi_x = \pi_x \phi_x = 0_A.$$

In [3] the groups G which are non-abelian and have a normal abelian subgroup A with G/A cyclic such that if p is a prime divisor of $|A|$ then $p^2 \not| |G/A|$; this includes the class $\mathcal{A}_p$. Fixing $x \in P - A$, the A-endomorphisms ϕ_x and π_x are shown to be nilpotent maps. For $a \in A$ define the numbers

$$a_\phi = \max\left\{i \geq 0 \,|\, \phi_x^i(a) \neq 0\right\} \qquad\qquad a_\pi = \max\left\{j \geq 0 \,|\, \pi_x^j(a) \neq 0\right\}.$$

Let $\widehat{a} = \phi_x^{a_\phi}(a)$ and $\overline{a} = \pi_x^{a_\pi}(a)$. A $\mathbb{Z}[\phi_x, \pi_x]$-module is completely decomposable into indecomposable submodules C which are identified uniquely by being open or closed and are generated by a special basis called a *chain basis* $\{a_1, \ldots, a_k\}$ with invariant ϕ and π weights

$$\{(a_1)_\phi, (a_1)_\pi, \ldots, (a_k)_\phi, (a_k)_\pi\}.$$

$(a_i)_\phi \neq 0$ if $2 \leq i \leq r$ and $(a_i)_\pi \neq 0$ if $1 \leq i < r$. An indecomposable module is *open* if $(a_1)_\phi = 0 = (a_k)_\pi$ and *closed* otherwise.

Szekeres' [3] characterization of $\mathbb{Z}[\phi_x, \pi_x]$-modules can be used to classify groups $P \in \mathcal{A}_p$ with an abelian maximal subgroup A, nilpotence class k, and rank m (that is $|P : \Phi(P)| = p^m$). p-Groups can have either $p+1$ or 1 abelian maximal subgroup and those with $p+1$ have been fully classified, hence we assume that A is unique.

As A is maximal we have $P = \langle x, A\rangle$ for any $x \in P - A$ and it can be shown that $x^p \in Z(P)$. If $x^p \in \mathrm{Im}(\pi_A)$ then there exists $z \in P - A$ such that $z^p = 1$; otherwise there exists a decomposition of A as a $\mathbb{Z}[\phi_x, \pi_x]$-module where $x^p = a_1^j$ where a_1 is the first element of an open chain basis and $1 \leq j < p$.

With a collection of chain bases for A and a suitable $x \in P - A$ it is easy to obtain a power-commutator presentation for P. The difficulty that remains in the classification of $\mathcal{A}_p$ is determining which presentations are isomorphic or nonisomorphic. We now focus our attention on how a collection of chain bases can be found.

Theorem 1.2. *The $(k-1)^{th}$-center, $Z_{k-1}(P)$, is properly contained in every abelian maximal subgroup A. The factor groups $A/Z_{k-1}(P)$ and $Z_i(P)/Z_{i-1}(P)$ for $2 \leq i < k$ are elementary abelian. Further, the sequence $\{u_1, \ldots, u_k\}$ defined $|A : Z_{k-1}(P)| = p^{u_k}$, $|\Omega_1(Z(P))| = p^{u_1}$, and $|Z_i(P) : Z_{i-1}(P)| = p^{u_i}$ satisfies*

$$m = u_1 \geq u_2 \geq \ldots \geq u_k \geq 1.$$

A suitable chain bases for A will correspond to a basis in the vector spaces $Z_i(P)/Z_{i-1}(P)$. There remains quite a lot of work to separate the isomorphism classes of $\mathcal{A}_p$, but important information concerning A and it's elements are in reach. What are the abelian invariants for an abelian maximal subgroup A?

Lemma 1.3. *For any $b \in A$ and $m > 0$ the element $\phi^m(b)$ has order p^r where $r = \left\lceil \frac{b_\phi - m - 1}{p-1} \right\rceil$.*

Further, b has order

$$\max\left(p^{\left\lceil\frac{b_\phi-1}{p-1}\right\rceil}, |\pi_A(b)|\right).$$

In order to determine the abelian invariants of A, it suffices to determine the abelian invariants of the $\mathbb{Z}[\phi_x, \pi_x]$-submodules of A for which A is a direct sum. Assume M is an irreducible $\mathbb{Z}[\phi_x, \pi_x]$-submodules of A with chain weights $\{(n_1, m_1), \ldots, (n_r, m_r)\}$, then M has rank $\sum_{i=1}^{r} \min(n_i, p-1)$.

1.1 Future Plans

I'm currently in a position to answer questions concerning the class $\mathcal{A}_p$ found in [**?**]: Classify the groups of $\mathcal{A}_p$ which are powerful p-groups, satisfy $|G'| = p^2$, or are of maximal class. Furthermore, I plan on studying the p-groups all of whose proper subgroups contain an abelian maximal subgroup.

2 Cyclic word subgroups

It is well known that elements in the commutator subgroup are not always commutators. In [4], I.D. Macdonald asks the question if a cyclic commutator subgroup is always generated by a commutator. He shows that this need not be the case. In fact, for every positive integer n, Macdonald provides examples of groups where the generators of the cyclic commutator subgroup are a product of not less than n commutators. But if the group is nilpotent or the commutator subgroup has infinite order, such a generating commutator always exists. Moreover, D.M. Rodney [5] has shown that the existence of a generating commutator for cyclic G', the commutator subgroup of the group G, does not necessarily imply that each element in G' is a commutator. In addition, Rodney gives sufficient conditions such that the existence of a generating commutator for cyclic G' implies that the set of commutators is equal to G'. For nilpotent groups or groups with an infinite commutator subgroup these conditions are satisfied.

These results by Macdonald [4] and Rodney [5] raise the following questions for an n-variable word $f(x_1, \ldots, x_n)$, the associated word subgroup $f(G)$ in a group G, and $F(G)$, the set of values of $f(x_1, \ldots, x_n)$ in G:

(1) Under what conditions on G do we have $f(G) = F(G)$?

(2) If $f(G)$ is cyclic, under what conditions is $f(G)$ generated by an element in $F(G)$?

(3) Is there a bound on the number of factors from $F(G)$ occurring in a generator of $f(G)$?

(4) If $f(G)$ is cyclic and generated by an element in $F(G)$, under what conditions do we have $f(G) = F(G)$?

In [6], the last three questions were addressed for $f(x_1, \ldots, x_n) = [x_1, \ldots, x_n]$, the normed commutator of weight n and the associated word subgroup $f(G) = G_n$, the corresponding term in the lower central series. The answers to these questions are very much in line with those given by Macdonald in [4] for the case of the commutator word of weight 2.

Luise-Charlotte Kappe and I gave an answer to the second and third question for the 1-variable word $f(x) = x^n$, where n is a positive integer. For the word $f(x)$ the last question always has an affirmative answer. It is well known that n-th powers of elements in a group do not necessarily form a subgroup. This question, with the exception of p-groups, has not found much attention in the literature until recently. Desmond MacHale in [7] shows that there are two groups of order 12, namely A_4, the alternating group on four letters, and $T = \langle a, b \mid a^3 = b^4 = 1, a^b = a^{-1} \rangle$, in which the squares do not form a subgroup, and that these groups are of minimal order with this property. In [8], the groups of minimal order for which the n-th powers do not form a subgroup were determined, where $n = 2^\alpha k$ with $\alpha = 0, 1, 2, 3$ and k odd.

To make the notions more precise, define $G^{(n)} = \{g^n \mid g \in G\}$ to be the set of elements in a group G raised to the n-th power, $n \in \mathbb{N}$, and $G^n = \langle g^n \mid g \in G \rangle$ as the subgroup generated by such elements.

Example 2.1. *[9] Let $q \mid p-1$ for primes p, q and let $\alpha, \beta \geq 2$. Choose an integer γ where $\gamma - 1 \nmid p^\alpha$ and $\gamma^q - 1 \mid p^\alpha$. For $n \in \{p^\tau q^\sigma \mid 1 \leq \tau < \alpha, 1 \leq \sigma < \beta\}$ the group*

$$G = \left\langle a, b \mid a^{p^\alpha} = b^{q^\beta} = 1, a^b = a^\gamma \right\rangle$$

is such that G^n is cyclic, but G^n can be generated by two n-th powers of elements of G but cannot be generated by a single n-th power.

The questions arises if there exist groups with a cyclic G^n requiring a product of more than two elements from $G^{(n)}$ for its generators. To our surprise, this was not the case.

Theorem 2.2. *[9] If G^n is a finite cyclic group, then there exists $a, b \in G$ such that $G^n = \langle a^n, b^n \rangle$.*

In pursuit of this goal we found some interesting structural properties of groups possessing a cyclic n-th power. Not surprisingly, these properties are in connection with n-abelian groups. A group is n-abelian if for every $a, b \in G$ we have that $(ab)^n = a^n b^n$.

Theorem 2.3. *[9] If G^n is an infinite cyclic group, then there exists an n-abelian $H \leq G$ such that $H^n = G^n$. Further, there exists $h \in H$ such that $G^n = H^n = \langle h^n \rangle$.*

Theorems 2.2 and 2.3 answer question (3). We turned our attention to question (2) and showed that a nilpotent group with a cyclic G^n can always be generated by a single element from $G^{(n)}$. This together with the fact that this is always true for infinite cyclic G^n, as mentioned above, gives the same sufficient conditions as established by Macdonald in [4] for the commutator word. In addition we obtain some sufficient conditions involving some arithmetic properties of n. This does not come as a surprise considering the nature of the problem.

2.1 Future Plans

As can be seen from the investigations in [6] and [4] on commutator words and our investigations here on the n-power word, it appears that the outcome very much depends on the nature of the word. One might ask the question to which other words Macdonald's investigations could be extended. What comes to mind are 2-variable words where the associated varieties are well understood, such as the 2-Engel word $[x, y, y]$ and the 3-abelian word $x^3y^3(xy)^{-3}$. Another class of groups comes to mind, namely groups with G'', the second derived subgroup, being cyclic, i.e. $f(G) = G''$ with $f(x, y, z, u) = [[x, y], [z, u]]$. It is perhaps easy to conjecture that in all those cases there exist groups with cyclic $f(G)$ not generated by a single element in $F(G)$. If the number of factors from $F(G)$ is bounded or not awaits further investigations.

3 Chermak-Delgado lattice of finite groups

The Chermak-Delgado measure was originally defined by A. Chermak and A. Delgado [10] as one in a family of functions from the subgroup lattice of a finite group into the positive integers.

Definition 3.1. *Let G be a finite group and $H \leq G$. The Chermak-Delgado measure of H (in G) is $m_G(H) = |H||C_G(H)|$.*

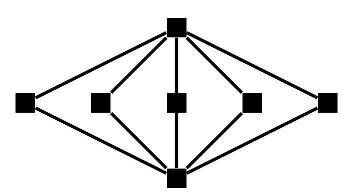
When G is clear from context we write simply $m(H)$. The maximum Chermak-Delgado measure possible in G is denoted by $m^*(G)$ and we use $\mathcal{CD}(G)$ to denote the set of all subgroups H with $m(H) = m^*(G)$. I. Martin Isaacs coined the term Chermak-Delgado measure, and proved that subgroups with maximal Chermak-Delgado measure form a modular sublattice in the subgroup lattice of the group [11]. Brewster and Wilcox then demonstrated that, for a direct product, this Chermak-Delgado lattice decomposes as the direct product of the Chermak-Delgado lattices of the factors [12], giving rise to the attention on the Chermak-Delgado lattice of p-groups (for a prime p).

Of particular note regarding $\mathcal{CD}(G)$ are the properties: If $H, K \in \mathcal{CD}(G)$ then $\langle H, K\rangle = HK$, $C_G(H) \in \mathcal{CD}(G)$, and also $C_G(C_G(H)) = H$. This latter property is typically referred to as the "duality property" of the Chermak-Delgado lattice. It is also known that the maximum subgroup in $\mathcal{CD}(G)$ is characteristic and the minimum subgroup is characteristic, abelian, and contains $\mathrm{Z}(G)$.

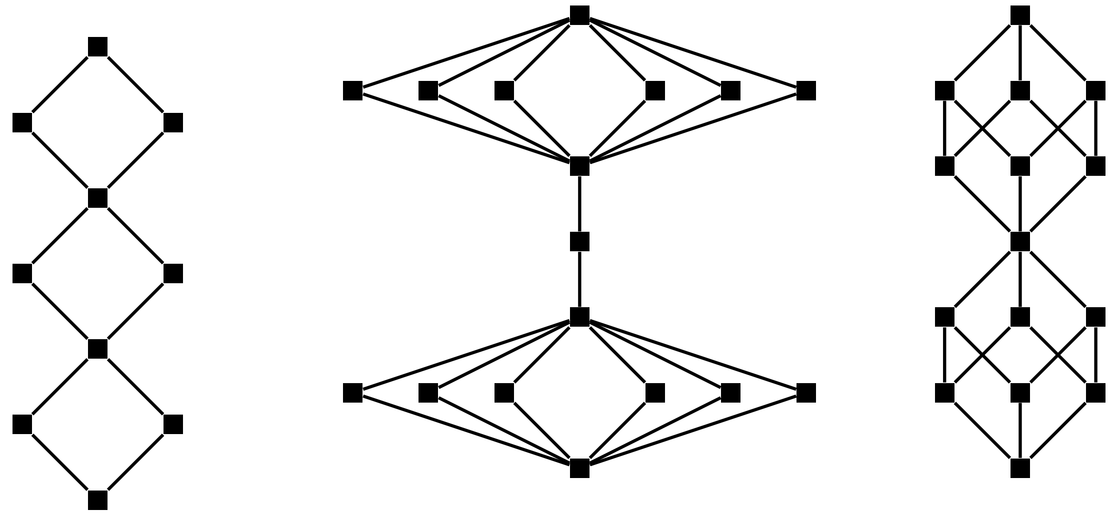
3.1 Chermake-Delgado Lattice Extension Theorems

The variety seen in the structure of the Chermak-Delgado lattices of p-groups seems inexhaustible; for example, there are many p-groups with a Chermak-Delgado lattice that is a single subgroup, a chain of arbitrary length, or a quasiantichain of width $p + 1$.

To describe the lattices constructed in later sections, we introduce the following terms for a positive integer n. A *quasiantichain of width n* will be denoted by $\mathcal{M}_n$. This is formed by an antichain of n elements joined at a unique maximum element and a unique minimum element. An *n-diamond* is a lattice with subgroups in the configuration of an n-dimensional cube. A *(uniform) n-string* is a lattice with n lattice isomorphic components, adjoined end-to-end so that the maximum of one component is identified with the minimum of the other component. A *mixed n-string* is a lattice with n components adjoined in the same fashion, though with at least one component not being lattice isomorphic to the remaining components. Examples illustrating this new terminology can be seen below.



$\mathcal{M}_7$



(a) A 3-string of 2-diamonds

(b) A mixed 3-string with two $\mathcal{M}_8$ components and a chain of length 2.

(c) A 2-string of 3-diamonds.

In [13] it is shown that certain p-groups N can be embedded into p-groups P whose Chermak-Delgado lattice is a mixed $2l+1$-string with center component lattice isomorphic to $\mathcal{CD}(N)$ and the other components are lattice isomorphic to either a m-diamond or $\mathcal{M}_{p+3}$. Moreover, the index between immediate descendants on the lattice of $\mathcal{CD}(P)$ is uniform and can be chosen to have order p^n for any $n \geq 2$.

Theorem 3.2. *Let N be a p-group such that $N \in \mathcal{CD}(N)$ and $\Phi(N) \leq Z(N)$. For any integers $n \geq 2$ and $m \geq 1$*

i) there exists a p-group P and a normal embedding of N into P, where $\mathcal{CD}(P)$ is a mixed 3-string with center component isomorphic to $\mathcal{CD}(N)$ and the remaining components being m-diamonds.

ii) there exists a p-group Q and a normal embedding of N into P, where $\mathcal{CD}(P)$ is a mixed 3-string with center component isomorphic to $\mathcal{CD}(N)$ and the remaining component $\mathcal{M}_{p+3}$.

In both of these groups, if two subgroups H and K with $H < K$ are in the Chermak-Delgado lattice and there does not exist a subgroup M in the Chermak-Delgado lattice such that $H < M < K$, then $|K : H| = p^n$.

3.2 Chermak-Delgado Simple p-Groups

Much of the previous work on the Chermak-Delgado measure has focused on determining which lattices arise as Chermak-Delgado lattices of finite groups. My work on extension theorems [13] implies that the shape of the $\mathcal{CD}$-lattice for a p-group of nilpotence class 2 can vary wildly. It is well known that the p-groups of nilpotence class 2 are a rich class of groups, so I've turned my attention to a class of p-groups with high nilpotence class and a restricted $\mathcal{CD}$-lattice.

Definition 3.3. *A group G is Chermak-Delgado Simple (or $\mathcal{CD}$-simple) if $\mathcal{CD}(G) = \{G, Z(G)\}$.*

In [12, 14] the Chermak-Delgado lattice was identified as a sublattice of the subnormal subgroup lattice. A trivial consequence is that the Chermak-Delgado lattice of any simple group G is $\{G, Z(G)\}$. Nonabelian $\mathcal{CD}$-simple groups do not possess large abelian subgroups.

Theorem 3.4. *If G is a nonabelian $\mathcal{CD}$-simple group then any abelian subgroup A is such that $\log_p(|A|) < \frac{n+z}{2}$ where $\log_p(|P|) = n$ and $\log_p(|Z(P)|) = z$.*

The Chermak-Delgado Lattice of S_n is $\{S_n, 1\}$ as the only subnormal subgroups are $\{S_n, A_n, 1\}$ and the measure of A_n is below half that of S_n. In [15] it is shown that smallest examples of non-abelian $\mathcal{CD}$-simple p-groups are of order p^5 for $p \neq 2$ and 2^6 otherwise.

The top and bottom elements of $\mathcal{CD}(G)$ are characteristic subgroups of G while the bottom element is always abelian. It follows that the Chermak-Delgado Measure of G can be obtained from the maximal measure of abelian characteristic subgroups of G; an observation that has been exploited in previous papers.

If $H, K \in \mathcal{CD}(G)$ are such that $H \leq K$ and there doesn't exist an $R \in \mathcal{CD}(G)$ with $H < R < K$, then H is normal in K [14]. From this observation we obtain a strategy in identifying $\mathcal{CD}$-simple groups:

Theorem 3.5. *If P and $Z(P)$ are the only normal subgroups of P in $\mathcal{CD}(P)$, then P is $\mathcal{CD}$-simple.*

Much of the focus on the Chermak-Delgado Lattice for p-groups has been placed on groups with small nilpotence class. A question of Ben Brewster:

Is there a $\mathcal{CD}$-simple group for every prime p and every nilpotence class?

The following construction is a 2-group of order 2^{3n} with nilpotence class $2n-1$, $n \geq 2$, which is $\mathcal{CD}$-simple.

Construction 3.6. *Let $n \geq 2$ and let F be the free group on the generators $\{a, b, c, d, e\}$. Define the relations*

$$R_n = \{a^2, b^4, c^{2^n}, d^{2^{n-1}}, e^{2^{n-2}}, [b,a]c^{-1}, [c,b]d^{-1}, [c,a]c^2, [c,d]c^4e^{-2}, [d,b]ed^2, [c,e], [d,e][c,d]^2\}$$

and define the group

$$G(n) = F/\langle R_n \rangle^F.$$

There exists a larger class of groups in which $G(n)$ belongs. Such groups are central extensions of maximal class p-groups.

Definition 3.7. *Let $\mathcal{S}_p$ be the class of 2-generator p-groups P for which there exists $n \geq 1$ with*

(i) $|P : Z(P)| = p^{2n+1}$

(ii) *The nilpotence class of P is $2n+1$.*

(iii) $C_P(Z_i(P)) = Z_{2n+2-i}(P)$ *for any $1 \leq i \leq 2n+1$.*

The lower p-exponent central series is defined $\lambda_0(P) = P$ and $\lambda_i(P) = \lambda_{i-1}(P)^p[\lambda_{i-1}(P), P]$ for $i \geq 1$. The lower p-exponent central series is the fastest descending p-exponent central series and is critically important to the study of p-groups. The lower p-exponent central series plays a key role in the *p-group generating algorithm*, an algorithm which generates a presentation for every finite p-group no two of which are isomorphic.

Theorem 3.8. *Let P be a non-abelian $\mathcal{CD}$-simple p-group and assume that d is the smallest integer such that $\lambda_d(P) \leq Z(P)$.*

- *$\lambda_d(P)$ is not cyclic.*
- *If $y \in \lambda_{d-1}(P) - Z(P)$ then $\Phi(P) \leq C_P(y)$.*
- *If P has rank f then*
$$\frac{\lambda_{d-1}(P)}{\lambda_{d-1}(P) \cap Z(P)}$$
is elementary abelian of rank r where $1 \leq r < f$.

The center of non-abelian $\mathcal{CD}$-simple p-groups are not cyclic. Furthermore, maximal class and extraspecial p-groups are not $\mathcal{CD}$-simple.

Theorem 3.9. *Assume that $P \in \mathcal{S}_p$ with $n \geq 3$ is such that $\Phi(P) = P'$. Then the grandparent, $P/\lambda_{2n-1}(P)$, is an element of $\mathcal{S}_p$.*

3.3 Future Plans

The unpredictability of centralizers factored through quotients of groups presents a major difficulty to the study of the Chermak-Delgado lattice and any understanding of how the lattice behaves through quotients would be valuable. I intend to study the invariants of the Chermak-Delgado lattice which are preserved through quotients of the lower exponent-p central series.

Conjecture 3.10. *A non-abelian $\mathcal{CD}$-simple p-group P which is a 2-generator group and has p-class c is such that $P/\lambda_{c-2}(P)$ is $\mathcal{CD}$-simple.*

MAGMA data seems to suggest that the above conjecture is true.

Conjecture 3.11. *The non-abelian $\mathcal{CD}$-simple p-groups which are 2-generator groups are exactly the members of $\mathcal{S}_p$.*

In my next investigation of $\mathcal{S}_p$, I will be calculating their irreducible characters and studying their normal subgroup lattice. From MAGMA evidence, the number of normal subgroups of the groups $G(n)$ is $2n^2 - n + 11$, that is $|G(30)| = 2^{60}$ has 791 normal subgroups. The groups of $\mathcal{S}_p$ are "middle" nilpotence class p-groups, which typically have much larger than quadratic growth in their normal subgroups.

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