Quiz 5 MATH 304 Section 3 Name: Solution

To receive credit, the solution to problems will be clearly presented. Partial credit will not be awarded for problems worked without comments. Unwanted work should be completely erased or clearly scratched out.

1. Let $X = \left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ be a subset of $\mathbb{R}^3$.

(i) Is X linearly independent? Justify your answer. — X LI iff rank$(X)=3$

(ii) Is X a basis for $\mathbb{R}^3$? If so, what is the value of $K_x\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$? — X basis iff X invertible

(i) $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so rank$(X)=3$ and X is LI.

(ii) As X is square with rank$(X)=$ # columns, X is invertible and X is a basis for $\mathbb{R}^3$.

$K_x\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$ is found by solving $\left[\begin{array}{ccc|c} 3 & -1 & 0 & -1 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2/13 \\ 0 & 1 & 0 & 7/13 \\ 0 & 0 & 1 & -10/13 \end{array}\right]$

$\Rightarrow K_x\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) = \frac{1}{13}\begin{bmatrix} -2 \\ 7 \\ -10 \end{bmatrix}$

2. For which values of a is $Y = \left\{ \begin{bmatrix} a^2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ a \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Y is a basis for $\mathbb{R}^3$ if and only if $\begin{bmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 2 & 1 \end{bmatrix}$ is invertible.

A matrix is invertible if it is square and rank $=$ # columns.

If $a=0$ then $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ has rank 2 ... not invertible.

If $a \neq 0$ then $\begin{bmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 2 & 1 \end{bmatrix}$ switch r_1, r_3, $r_3 = R_3 - a^2 R_1$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & a & 0 \\ 0 & -2a^2 & 1-a^2 \end{bmatrix}$ $r_3 = R_3 + 2a R_2$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & a & 0 \\ 0 & 0 & 1-a^2 \end{bmatrix}$ — rank 3 only if $1-a^2 \neq 0$

$\Rightarrow a \neq \pm 1$

$\{a \mid a \neq 0, 1, -1\}$