True + False

#1

a) T
b) F
c) T
d) F

e) T
f) T
g) T
h) F

i) T
j) T
l) T
m) F

n) T
o) T
p) F
q) T

r) T
s) F
t) T

#12

a) T
~~b) T~~
c) F
d) T

g) F
h) F
j) T

l) F
m) T
n) F
o) T

(2) $\mathcal{W}$ cannot span $\mathbb{R}^3$, <u>not</u> a basis.

$\mathcal{X}$ cannot be LI, <u>not</u> a basis.

$\mathcal{Y}$ in RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\mathcal{Y}$ is invertible as a matrix so $\mathcal{Y}$ is a basis for $\mathbb{R}^3$

$\mathcal{Z}$ in RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (same for $\mathcal{Z}$

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(3) $M = \begin{bmatrix} 1 & 1 & 5 & 1 & 4 \\ 2 & -1 & 1 & 2 & 2 \\ 3 & 0 & 6 & 0 & -3 \end{bmatrix}$ in RREF $\begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

A basis for col(M) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

(notice M is onto!)

A basis for nul(M) is found by taking the parametric solution of $M\vec{x} = \vec{0}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_5 - 2x_3 \\ -2x_5 - 3x_3 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

A basis for nul(M) is $\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$

4) N in RREF $\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

A basis for col(N) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

Solutions to $N\vec{x} = \vec{0}$ are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

A basis for nul(N) is

$\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

A basis for row(N) is $\left\{ \begin{bmatrix} 2 & 1 & 4 & -1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 5 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -7 & 0 & 1 \end{bmatrix} \right\}$

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5) X as a matrix in RREF $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

a) since the rank of X is 3,
$\dim(\mathrm{col}(X)) = 3$. $\mathbb{R}^3 = \mathrm{Span}(X)$

b) As the rank of X is less than 4, X is LD
$\vec{0} = \vec{v}_1 - \vec{v}_2 + \vec{v}_3$

c) $\vec{v}_3 = -\vec{v}_1 + \vec{v}_2$ or $\vec{v}_1 = \vec{v}_2 - \vec{v}_3$

d) $X' = (\vec{v}_2, \vec{v}_3, \vec{v}_4)$ is a basis for $\mathbb{R}^3$ as
$\rightarrow$ X' is LI (rank is 3)
$\rightarrow$ $\mathrm{Span}(X') = \mathrm{span}(X) = \mathbb{R}^3$ (span preservation)

6) Y as a matrix

(a)

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 6 \\ 0 & -2 & -3 \\ 1 & 6 & 5 \end{bmatrix} \begin{matrix} r_1 = \frac{1}{2}R_1 \\ r_2 = \frac{1}{2}R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -2 & -3 \\ 1 & 6 & 5 \end{bmatrix} \begin{matrix} r_3 = R_2 + R_3 \\ r_4 = R_4 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} r_4 = R_4 - 2R_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

rank $= 3$
$= \#$ elements
$\therefore Y$ is LI

(b) let $\vec{w}_4 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, $\vec{w}_4 \notin \text{Span}(Y)$ if $[Y \,\vdots\, \vec{w}_4]$ is inconsistent

$$\left[\begin{array}{ccc|c} 2 & 4 & 2 & a \\ 0 & 4 & 6 & b \\ 0 & -2 & -3 & c \\ 1 & 6 & 5 & d \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & a/2 \\ 0 & 2 & 3 & b/2 \\ 0 & -2 & -3 & c \\ 1 & 6 & 5 & d \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & a/2 \\ 0 & 2 & 3 & b/2 \\ 0 & 0 & 0 & c + b/2 \\ 0 & 4 & 4 & d - a/2 \end{array}\right]$$

if $c + b/2 \neq 0$ then inconsistent

take $\vec{w}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

(c) $Y \cup \{\vec{w}_4\}$ is LI by independence extension.
Therefore, $\dim(\text{span}(Y \cup \{\vec{w}_4\})) = 4$
so $Y \cup \{\vec{w}_4\}$ spans $\mathbb{R}^4$!

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(7) $q(x) = ap_1(x) + bp_2(x) + cp_3(x)$

$\Rightarrow 5 + 9x + 5x^2 = (2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2$

$\Rightarrow 5 = 2a + b + 3c$
$9 = a - b + 2c$
$5 = 4a + 3b + 5c$

solve $\left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 1 & -1 & 2 & 9 \\ 4 & 3 & 5 & 5 \end{array}\right]$

$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$

Solution $\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$ (notice $\{p_1, p_2, p_3\}$ forms a basis for $\mathbb{P}_2$)

(8) $\mathcal{Y} = \{1+2x-x^2,\ 3+x^2,\ 5+4x-x^2,\ -2+2x-2x^2\}$

Using the standard basis for $\mathbb{P}_2$, $\mathcal{X} = \{1, x, x^2\}$ we have

$$K_{\mathcal{X}}(\mathcal{Y}) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} \right\}$$

Since $K_{\mathcal{X}}$ is an isomorphism, $\mathcal{Y}$ spans $\mathbb{P}_2$ if and only if $K_{\mathcal{X}}(\mathcal{Y})$ spans $\mathbb{R}^3$.

$K_{\mathcal{X}}(\mathcal{Y})$ in RREF is $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

so $\operatorname{rank}(K_{\mathcal{X}}(\mathcal{Y})) = 2$ and $\mathcal{Y}$ does <u>not</u> span $\mathbb{P}_2$.

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(9) (a) $\mathcal{X}_1$ is a basis for $\mathbb{P}_2$ only if $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

in RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathcal{X}_1$ is a basis for $\mathbb{P}_2$.

(b) $\mathcal{X}_2$ is too small to span $\mathbb{P}_2$, so it isn't a basis.

(c) $\mathcal{X}_3$ is a basis for $\mathbb{P}_2$ only if $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

in RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathcal{X}_3$ is a basis for $\mathbb{P}_2$.

(d) $\mathcal{X}_4$ is too large to be linearly independent, so it isn't a basis for $\mathbb{P}_2$.

(10) (a) $\left[T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \right] = \begin{bmatrix}1 & 3 & 4\\ 1 & 0 & -7\end{bmatrix}$

(b) $T\left(\begin{bmatrix}1\\3\\8\end{bmatrix}\right) = 1T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 3T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + 8T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$

$= \begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}3\\0\end{bmatrix} + 8\begin{bmatrix}4\\-7\end{bmatrix} = \begin{bmatrix}42\\-55\end{bmatrix}$

(c) $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1+3x_2+4x_3\\ x_1-7x_3\end{bmatrix}$

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(13) (a) $\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ (c) $\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

(d) $\begin{bmatrix}\cos(\pi/6) & -\sin(\pi/6)\\ \sin(\pi/6) & \cos(\pi/6)\end{bmatrix}$ (e) $\begin{bmatrix}\cos(\pi/3) & \sin(\pi/3)\\ -\sin(\pi/3) & \cos(\pi/3)\end{bmatrix}$

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(14) (a) $\begin{bmatrix}0\\1\end{bmatrix} = -\frac{2}{3}\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{3}\begin{bmatrix}2\\7\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix} = \frac{7}{3}\begin{bmatrix}1\\2\end{bmatrix} - \frac{2}{3}\begin{bmatrix}2\\7\end{bmatrix}$

(coefficients found by solving $\left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 2 & 7 & 0 & 1\end{array}\right]$)

(b) $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{7}{3}T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - \frac{2}{3}T\left(\begin{bmatrix}2\\7\end{bmatrix}\right) = \frac{7}{3}\begin{bmatrix}1\\3\\5\end{bmatrix} - \frac{2}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}9\\19\\33\end{bmatrix}$

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = -\frac{2}{3}T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + \frac{1}{3}T\left(\begin{bmatrix}2\\7\end{bmatrix}\right) = -\frac{2}{3}\begin{bmatrix}1\\3\\5\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-3\\-5\\-9\end{bmatrix}$

(c) $\begin{bmatrix}3 & -1\\ 19/3 & -5/3\\ 11 & -3\end{bmatrix}$

15. (a) $A = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 7 & -1 \\ 1 & 1 & 5 & 0 \end{pmatrix}$

(b) $A \longrightarrow \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ $c_4 = 0$, $c_3 = t \in \mathbb{R}$, $c_2 = -7t$, $c_1 = 2t$ $\Rightarrow \underline{v} = \begin{pmatrix} 2 \\ -7 \\ 1 \\ 0 \end{pmatrix} \in \ker(G)$

(c) 3 pivot columns $\Rightarrow$ rank $A = 3 \Rightarrow G$ onto

Let $X = (\underline{v_1}, \underline{v_2}, \underline{v_3})$

16. $A = [\underline{v_1}\ \underline{v_2}\ \underline{v_3}] = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 4 & 2 \\ 4 & 7 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$

(a) rank $A = 2 < 3 \Rightarrow X$ does not span $\mathbb{R}^3$.
(A not onto)

(b) $\text{Span}(X) = \left\langle \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} \right\rangle$

Let $B := (a\ b\ c) : \mathbb{R}^3 \to \mathbb{R}$ with $\ker B = \text{Span}(X)$

(recall $\ker(B) = \{ \underline{x} \in \mathbb{R}^3 : B\underline{x} = 0 \}$
and $3 = \dim \ker(B) + \dim \operatorname{im}(B)$)

By $(a\ b\ c)\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 0$ and $(a\ b\ c)\begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} = 0$

$\Rightarrow$ $a + 3c = 0$

(c) X is not linearly independent, because
rank $A = 2 < 3$ (not 1-1)

(d) By the RREF $c_3 = t \in \mathbb{R}$, $c_2 = -t/3$, $c_1 = -2t/3$
$\Rightarrow -2\underline{v_1} - v_2 + 3\underline{v_3} = \underline{0}$

17. Let $\mathcal{Y} = (\underline{w_1}, \underline{w_2}, \underline{w_3}, \underline{w_4})$.

$$A = [\underline{w_1}\ \underline{w_2}\ \underline{w_3}\ \underline{w_4}] = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 5 & -5 & -7 & -3 \\ -7 & 9 & 11 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 5 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) $\operatorname{rank} A = 2 < 4$ (A not onto) $\Rightarrow \mathcal{Y}$ does not span $\mathbb{R}^4$

(b) $\underline{e_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \notin \operatorname{Span}(\mathcal{Y})$ because

$[A | \underline{e_4}]$ is inconsistent

(c) $\mathcal{Y}$ is not linearly independent because $\operatorname{rank} A = 2 < 4$

(A not 1-1)

(d) By the REF :
$c_4 = t \in \mathbb{R}$
$c_3 = s \in \mathbb{R}$
$c_2 = \frac{1}{5}(-3s - 2t)$
$c_1 = \frac{1}{5}(4s + t)$

For $s = t = 1 \Rightarrow c_1 = 1$, $c_2 = -1$

$\Rightarrow \underline{w_1} - \underline{w_2} + \underline{w_3} + \underline{w_4} = \underline{0}$

18. (a) linearly dependent ($\underline{0} \in X_a$)

(b) linearly dependent ($(12, 4, -8) = 4 \cdot (3, 1, -2)$)

(c) linearly independent : the two vectors are not multiple of each other

(d) linearly dependent ($(3, -9, -6) = -\frac{3}{2}(-2, 6, 4)$)

(e) linearly dependent (4 vectors in $\mathbb{R}^3$)