Practice problems for Test 1

1. Label the following statements as TRUE or FALSE.

   (a) If A is a matrix with no zero columns then its reduced row echelon form may have a zero column.

   (b) A matrix may have two different row echelon forms.

   (c) A matrix may have two different reduced row echelon forms.

   (d) A system of linear equations with fewer unknowns than equations must have infinitely many solutions or none.

   (e) A system of linear equations with more unknowns than equations must have infinitely many solutions or none.

   (f) Any homogeneous system of linear equations with more variables than equations has infinitely many solutions.

   (g) The rank of a matrix is equal to the number of nonzero rows in its reduced row echelon form.

   (h) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.

   (i) Elementary row operations do not necessarily preserve rank.

   (j) The rank of an $m \times n$ matrix is at most the largest of the integers m and n.

   (k) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.

   (l) If C is the coefficient matrix of a system of linear equations, $A = [C \,|\, \vec{b}]$ is the augmented matrix of the system, and the rank of C is less than the rank of A then the system is inconsistent.

   (m) If the augmented matrix $[A \,|\, \vec{b}]$ has reduced row echelon form $[A' \,|\, \vec{b}]$, then A' is the reduced row echelon form of A.

   (n) If A and B are both $n \times n$ matrices, then $AB = BA$.

   (o) If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then $(AB)^T = A^T B^T$.

2. Use the row reduction procedure described in class (or in the book) to find the reduced row echelon form of the matrix

$$N = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \\ -2 & -5 & 8 & 0 & -17 \end{bmatrix}.$$

3. Suppose

$$M = \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 4 & 3 & 1 \\ 0 & 3 & -6 & 8 & 1 & -2 \\ 1 & 1 & -2 & 1 & 3 & 2 \\ 1 & -1 & 2 & -4 & 2 & 3 \end{array}\right]$$

is the augmented matrix of a system of four linear equations in the variables x_1, x_2, x_3, x_4, x_5. After a lengthy calculation we find the reduced row echelon form of M is

$$R = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Which variables are the basic variables?

(b) Which variables are the free variables?

(c) What are the pivot columns of M?

(d) What is the rank of M?

(e) If the system is consistent write the solution in parametric form.

4. Solve the following system of linear equations.

$$\begin{aligned} 2x_1 + x_2 + x_3 &= -1 \\ x_1 \qquad - x_3 &= 0 \\ 6x_1 + 2x_2 + x_3 &= 1 \end{aligned}$$

5. Solve the following system of linear equations.

$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= -2 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 + x_2 - x_3 \qquad &= 0 \end{aligned}$$

6. Solve the following system of linear equations.

$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= 1 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 + x_2 - x_3 \qquad &= 0 \end{aligned}$$

7. Solve the following system of linear equations.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 &= 17 \\ x_1 + x_2 + x_3 + x_4 - 3x_5 &= 6 \\ x_1 + x_2 + x_3 + 2x_4 - 5x_5 &= 8 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 &= 14 \end{aligned}$$

8. Let

$$A = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ -1 & 4 & -2 & t \\ 8 & -11 & 13 & -1 \end{array}\right]$$

be the augmented matrix of a system of linear equations. For what value(s) of t is the system inconsistent?

9. Let

$$T = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \end{bmatrix}.$$

Suppose that T is used to define a function in the usual way.

(a) Compute the rank of T.

(b) What is the domain of T?

(c) What is the codomain of T?

(d) Is T onto? Why or why not?

(e) Is T one-to-one? Why or why not?

10. Let $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the function defined by the formula

$$F(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3, x_1 - x_3).$$

(a) Show that F is a one-to-one correspondence.

(b) Find a formula for the inverse, F^{-1}, of F.

11. Let $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the function defined by the formula

$$G(x_1, x_2, x_3) = (-3x_1 + 6x_2 + 9x_3, x_1 - 2x_2 - 2x_3, 2x_1 - 4x_2 - 3x_3).$$

(a) Find a point in $\mathbf{R}^3$ which is *not* in the image of G.

(b) Find two different points in $\mathbf{R}^3$ whose image under G is $(-3, 1, 2)$.

12. Consider the matrices

$$A = \begin{bmatrix} -1 & 5 & 2 \\ 7 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 4 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -4 & 0 \\ 1 & 4 \end{bmatrix} \qquad E = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \qquad F = \begin{bmatrix} -1 & 5 \\ 4 & -3 \\ 1 & 0 \end{bmatrix}$$

For each of the following operations, either do the indicated calculations or explain why it is not defined.

(a) DC (b) $B - 2A$ (c) BF (d) $3C - E$ (e) ED

(f) $B((A^T + F)D)$ (g) $(A + B)C$ (h) FA (i) $A(FE)$ (j) $(F(C + D)^T + B^T)E$

13. (Here is a slightly challenging problem.) A box containing pennies, nickels, and dimes has 13 coins with a total value of 83 cents. How many coins of each type are in the box?