Homework 20 MATH 304 Section 3

Assigned: Friday, November 14.
Potentially Collected: Friday, November 21.

1. Let V be the vector space with basis $S = (1, t, e^t, te^t)$ and note that V is a subspace of the vector space of differentiable real functions. Let $L : V \to V$ be the linear transformation where $L(f(t)) = f'(t)$ for any differentiable function f. Find $_SL_S$.

2. Let $F : \mathbb{P}_1 \to \mathbb{P}_2$ be the linear transformation where $F(p(x)) = xp(x) + p(0)$ for any degree one polynomial p. Consider the bases $S = (x, 1)$ and $R = (x+1, x-1)$ for $\mathbb{P}_2$ and $A = (x^2, 1, x)$ and $B = (x^2+1, x-1, x+1)$ for $\mathbb{P}_3$. $E_1 = (1, x)$ $E_2 = (1, x, x^2)$

   (a) Find the standard matrix for F and then calculate $_AF_S$ and $_BF_R$ using change of basis matrices.

   (b) Calculate $F(-3x-3)$ for each of the three matrices of (a).

3. Let $L : M_{22} \to M_{22}$ be the linear transformation between the vector space of 2×2 real matrices where $L(A) = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A\right)$ for any 2×2 real matrix A. Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

   (a) Show that S and T are bases for M_{22}.

   (b) Find $_SL_S$, $_TL_T$, $_SL_T$, and $_TL_S$.

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① $_SL_S = [K_S L(1)\ \ K_S L(t)\ \ K_S L(e^t)\ \ K_S L(te^t)]$

$= [K_S(0)\ \ K_S(1)\ \ K_S(e^t)\ \ K_S(te^t + e^t)]$

$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

let $f(t) = 1 + t + e^t + te^t$

$L(f(t)) = 0 + 1 + e^t + te^t + e^t$

by derivatives.

$L(f(t)) = K_S^{-1}(_SL_S)K_S(f(t))$

$= K_S^{-1}\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$

$= K_S^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right) = 1 + 2e^t + te^t$

② $_{E_2}F_{E_1} = [K_{E_2}F(1)\ \ K_{E_2}F(x)]$

$= [K_{E_2}(x+1)\ \ K_{E_2}(x^2)]$

$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$_A I_{E_2} = [K_A(1) \;\; K_A(x) \;\; K_A(x^2)]$

solve $\vec{w} = a_1(x^2) + a_2(1) + a_3(x) = a_2 + a_3 x + a_1 x^2$

to find $K_A(\vec{w}) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$. Solve $\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

$\hookrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$

$_A I_{E_2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$_B I_{E_2} = [K_B(1) \;\; K_B(x) \;\; K_B(x^2)]$

solve $\vec{w} = a_1(x^2+1) + a_2(x-1) + a_3(x+1) = (a_1 - a_2 + a_3) + (a_2 + a_3)x + (a_1)x^2$

Solve $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$ to obtain $_B I_{E_2} = \begin{bmatrix} 0 & 0 & 1 \\ -.5 & .5 & .5 \\ .5 & .5 & -.5 \end{bmatrix}$

$_{E_1} I_S = [K_{E_1}(x) \;\; K_{E_1}(1)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$_{E_1} I_R = [K_{E_1}(x+1) \;\; K_{E_1}(x-1)] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$_A F_S = (_A I_{E_2})(_{E_2} F_{E_1})(_{E_1} I_S) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

$_B F_R = (_B I_{E_2})(_{E_2} F_{E_1})(_{E_1} I_R) = \begin{bmatrix} 0 & 0 & 1 \\ -.5 & .5 & .5 \\ .5 & .5 & -.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ -.5 & .5 & .5 \\ .5 & .5 & -.5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ .5 & .5 \\ .5 & -1.5 \end{bmatrix}$

(b) ${}_A F_S = K_A F K_S^{-1} \Rightarrow F = K_A^{-1}({}_A F_S) K_S$

so $F(-3x-3) = K_A^{-1}({}_S F_A) K_S(-3x-3)$

$= K_A^{-1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right) = K_A^{-1}\left(\begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}\right)$

$= -3(x^2) - 3(1) - 3(x) = -3 - 3x - 3x^2$

${}_B F_R = K_B F K_R^{-1} \Rightarrow F = K_B^{-1}({}_B F_R) K_R$

So $F(-3x-3) = K_B^{-1}({}_B F_R) K_R(-3x-3)$

$= K_B^{-1}({}_B F_R)\begin{bmatrix} -3 \\ 0 \end{bmatrix}$

$= K_B^{-1}\left(\begin{bmatrix} -3 \\ -1.5 \\ -1.5 \end{bmatrix}\right) = -3 - 3x - 3x^2$

$-3x-3 = a(x+1) + b(x-1)$
$= (a+b)x + (a-b)$
solve $\left[\begin{array}{cc|c} 1 & 1 & -3 \\ 1 & -1 & -3 \end{array}\right]$

${}_{E_2} F_{E_1} = K_{E_2} F K_{E_1}^{-1} \Rightarrow F = K_{E_2}^{-1}({}_{E_2} F_{E_1}) K_{E_1}$

So $F(-3x-3) = K_{E_2}^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} K_{E_1}(-3x-3)$

$= K_{E_2}^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix} = K_{E_2}^{-1}\left(\begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}\right)$

$= -3 - 3x - 3x^2$

(3) Is S LI? solve $\begin{bmatrix}0&0\\0&0\end{bmatrix} = a\begin{bmatrix}1&0\\0&0\end{bmatrix} + b\begin{bmatrix}0&1\\0&0\end{bmatrix} + c\begin{bmatrix}0&0\\1&0\end{bmatrix} + d\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$$\begin{bmatrix}0&0\\0&0\end{bmatrix} = \begin{bmatrix}a&b\\c&d\end{bmatrix} \Rightarrow \begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

only trivial solution

$$\text{Span}(S) = \left\{ \begin{bmatrix}a&b\\c&d\end{bmatrix} \middle| a,b,c,d \in \mathbb{R} \right\} = M_{22}$$

Therefore, S is a basis for M_{22}.

Is T LI? solve $\begin{bmatrix}0&0\\0&0\end{bmatrix} = a\begin{bmatrix}1&0\\0&1\end{bmatrix} + b\begin{bmatrix}1&1\\0&0\end{bmatrix} + c\begin{bmatrix}1&0\\1&0\end{bmatrix} + d\begin{bmatrix}0&1\\0&0\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}0&0\\0&0\end{bmatrix} = \begin{bmatrix}a+b+c & b+d\\ c & a\end{bmatrix}$$

$\Rightarrow$ $c=0$, $a=0$, $b=0$, $d=0$ only trivial solution

As $\dim(\text{span}(T)) = 4 = \dim(M_{22})$

we have that T is a basis for M_{22}.

$${}_S L_S = \left[K_S L\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right) \; K_S L\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) \; K_S L\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right) \; K_S L\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) \right]$$

$$= \left[K_S\left(\begin{bmatrix}1&0\\3&0\end{bmatrix}\right) \; K_S\left(\begin{bmatrix}0&1\\0&3\end{bmatrix}\right) \; K_S\left(\begin{bmatrix}2&0\\4&0\end{bmatrix}\right) \; K_S\left(\begin{bmatrix}0&2\\0&4\end{bmatrix}\right) \right]$$

$$= \begin{bmatrix}1&0&2&0\\0&1&0&2\\3&0&4&0\\0&3&0&4\end{bmatrix}$$

$${}_TL_T = \left[K_T L\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) \; K_T L\left(\begin{bmatrix}1&1\\0&0\end{bmatrix}\right) \; K_T L\left(\begin{bmatrix}1&0\\1&0\end{bmatrix}\right) \; K_T L\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) \right]$$

$$= \left[K_T\left(\begin{bmatrix}1&2\\3&4\end{bmatrix}\right) \; K_T\left(\begin{bmatrix}1&1\\3&3\end{bmatrix}\right) \; K_T\left(\begin{bmatrix}3&0\\7&0\end{bmatrix}\right) \; K_T\left(\begin{bmatrix}0&1\\0&3\end{bmatrix}\right) \right]$$

$$= \begin{bmatrix} 4 & 3 & 0 & 3 \\ -6 & -5 & -4 & -3 \\ 3 & 3 & 7 & 0 \\ 8 & 6 & 4 & 4 \end{bmatrix}$$

$${}_SL_T = \left[K_S L\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) \; K_S L\left(\begin{bmatrix}1&1\\0&0\end{bmatrix}\right) \; K_S L\left(\begin{bmatrix}1&0\\1&0\end{bmatrix}\right) \; K_S L\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) \right]$$

$$= \left[K_S\left(\begin{bmatrix}1&2\\3&4\end{bmatrix}\right) \; K_S\left(\begin{bmatrix}1&1\\3&3\end{bmatrix}\right) \; K_S\left(\begin{bmatrix}3&0\\7&0\end{bmatrix}\right) \; K_S\left(\begin{bmatrix}0&1\\0&3\end{bmatrix}\right) \right]$$

$$= \begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 3 & 7 & 0 \\ 4 & 3 & 0 & 3 \end{bmatrix}$$

$${}_TL_S = \left[K_T L\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right) \; K_T L\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) \; K_T L\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right) \; K_T L\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) \right]$$

$$= \left[K_T\left(\begin{bmatrix}1&0\\3&0\end{bmatrix}\right) \; K_T\left(\begin{bmatrix}0&1\\0&3\end{bmatrix}\right) \; K_T\left(\begin{bmatrix}2&0\\4&0\end{bmatrix}\right) \; K_T\left(\begin{bmatrix}0&2\\0&4\end{bmatrix}\right) \right]$$

$$= \begin{bmatrix} 0 & 3 & 0 & 4 \\ -2 & -3 & -2 & -4 \\ 3 & 0 & 4 & 0 \\ 2 & 4 & 2 & 6 \end{bmatrix}$$