Homework 19 MATH 304 Section 3

Assigned: Wednesday, November 12.
Potentially Collected: Wednesday, November 19.

1. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\1\\0\end{bmatrix} \quad L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\0\\1\end{bmatrix} \quad L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\\1\end{bmatrix}$$

(a) Let E be the standard basis of $\mathbb{R}^3$ and find ${}_EL_E$.

(b) Let $R = \left(\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\2\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$ and find ${}_EL_R$, ${}_RL_E$, and ${}_RL_R$ through the definition or through change of basis matrices.

(c) Find $L\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$ using each of the four matrices found in (a) and (b).

$${}_EL_E = \left[K_E L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \; K_E L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \; K_E L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right]$$

(a) $$= \left[K_E\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) \; K_E\left(\begin{bmatrix}2\\0\\1\end{bmatrix}\right) \; K_E\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)\right] = \begin{bmatrix}1&2&1\\1&0&0\\0&1&1\end{bmatrix}$$

$${}_RI_E = \left[K_R\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \; K_R\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \; K_R\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right]$$

solve $\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&2&1&0&1&0\\1&2&-1&0&0&1\end{array}\right]$ $\qquad {}_RI_E = \begin{bmatrix}1&0&0\\-.25&.25&.25\\.5&.5&-.5\end{bmatrix}$

$${}_EI_R = \left[K_E\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) \; K_E\left(\begin{bmatrix}0\\2\\2\end{bmatrix}\right) \; K_E\left(\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)\right] = \begin{bmatrix}1&0&0\\0&2&1\\1&2&-1\end{bmatrix}$$

$${}_EL_R = ({}_EL_E)({}_EI_R) = \begin{bmatrix}2&6&1\\1&0&0\\1&4&0\end{bmatrix}$$

$${}_RL_E = ({}_RI_E)({}_EL_E) = \begin{bmatrix}1&2&1\\0&-1/4&0\\1&1/2&0\end{bmatrix}$$

$${}_RL_R = ({}_RI_E)({}_EL_R) = ({}_RL_E)({}_EI_R) = \begin{bmatrix}2&6&1\\0&-1/2&-1/4\\1&1&1/2\end{bmatrix}$$

c $\quad L = K_E^{-1}({}_E L_E) K_E$

$$= K_R^{-1}({}_R L_E) K_E$$

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$$= K_E^{-1}({}_E L_R) K_R$$

$$K_E^{-1}({}_E L_E) K_E\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = K_E^{-1}\left(\begin{bmatrix}1&2&1\\1&0&0\\0&1&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = K_E^{-1}\left(\begin{bmatrix}8\\1\\5\end{bmatrix}\right)$$
$$= \begin{bmatrix}8\\1\\5\end{bmatrix}$$

$$K_R^{-1}({}_R L_E) K_E\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = K_R^{-1}\left(\begin{bmatrix}1&2&1\\0&-1/4&0\\1&1/2&0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = K_R^{-1}\left(\begin{bmatrix}8\\-1/2\\2\end{bmatrix}\right)$$
$$= 8\begin{bmatrix}1\\0\\1\end{bmatrix} + (-1/2)\begin{bmatrix}0\\2\\2\end{bmatrix} + (2)\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}8\\2/2\\10/2\end{bmatrix} = \begin{bmatrix}8\\1\\5\end{bmatrix}$$

$$K_R^{-1}({}_R L_R) K_R\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) \longleftarrow \boxed{K_R\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) \text{ solve } \left[\begin{array}{ccc|c}1&0&0&1\\0&2&1&2\\1&2&-1&3\end{array}\right]}$$

$$= K_R^{-1}\left(\begin{bmatrix}2&6&1\\0&-1/2&1/4\\1&1&1/2\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$$
$$= K_R^{-1}\left(\begin{bmatrix}8\\-1/2\\2\end{bmatrix}\right) = 8\begin{bmatrix}1\\0\\1\end{bmatrix} + (-1/2)\begin{bmatrix}0\\2\\2\end{bmatrix} + (2)\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}8\\1\\5\end{bmatrix}$$

$$K_E^{-1}({}_E L_R) K_R\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = K_E^{-1}\left(\begin{bmatrix}2&6&1\\1&0&0\\1&4&0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = K_E^{-1}\left(\begin{bmatrix}8\\1\\5\end{bmatrix}\right)$$
$$= \begin{bmatrix}8\\1\\5\end{bmatrix}$$