Homework 18 MATH 304 Section 3

Assigned: Monday, November 10.
Potentially Collected: Monday, November 17.

1. Define the linear transformation $D : \mathbb{P}_2 \to \mathbb{P}_2$ where $D(p(x)) = p'(x)$. Let $Y = (1, x, x^2)$ and $A = (-1+x, 1+x)$.
   (a) Is A a basis for $\mathbb{P}_2$? Find a basis Z for $\mathbb{P}_2$ which contains A.
   (b) Find ${}_YD_Y$.
   (c) Find ${}_ZD_Z$, ${}_YD_Z$, and ${}_ZD_Y$ through the definition.
   (d) Find the change of basis matrices ${}_YI_Z$ and ${}_ZI_Y$. Use the change of basis matrices and ${}_YD_Y$ to find ${}_ZD_Z$, ${}_YD_Z$, and ${}_ZD_Y$.

(a) A is not a basis as $\dim(\mathrm{Span}(A)) = 2$ and $\dim(\mathbb{P}_2) = 3$.

let $Z = (-1+x, 1+x, x^2)$ [Note: any polynomial of the form $a_0 + a_1x + a_2x^2$ w/ $a_2 \neq 0$ will work (Independence Extension)]

(b) ${}_yD_y = [K_yD(1) \quad K_yD(x) \quad K_yD(x^2)] = [K_y(0) \quad K_y(1) \quad K_y(2x)]$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) ${}_yD_z = [K_yD(-1+x) \quad K_yD(1+x) \quad K_yD(x^2)] = [K_y(1) \quad K_y(1) \quad K_y(2x)]$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

${}_zD_y = [K_zD(1) \quad K_zD(x) \quad K_zD(x^2)] = [K_z(0) \quad K_z(1) \quad K_z(2x)]$

the map $K_z(\vec{w}) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $\vec{w} = a(-1+x) + b(1+x) + c(x^2)$

$= (-a+b) + (a+b)x + (c)x^2$

solve $\left[\begin{array}{ccc|c} -1 & 1 & 0 & \\ 1 & 1 & 0 & \vec{w} \\ 0 & 0 & 1 & \end{array}\right]$

${}_zD_y$ is the solution to $\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$

So

${}_zD_y = \begin{bmatrix} 0 & -1/2 & 1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$ {}_zD_z = [K_z D(-1+x) \quad K_z D(1+x) \quad K_z(x^2)] $$
$$ = [K_z(1) \quad K_z(1) \quad K_z(2x)] $$

the solution to $\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$ is ${}_zD_z$

$$ {}_zD_z = \begin{bmatrix} -1/2 & -1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} $$

(d) $$ {}_yI_z = [K_y(-1+x) \quad K_y(1+x) \quad K_y(x^2)] $$
$$ = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}_zI_y = [K_z(1) \quad K_z(x) \quad K_z(x^2)] $$
$$ = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

solve $\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$$ {}_zD_z = ({}_zI_y)({}_yD_y)({}_yI_z) $$
$$ {}_zD_y = ({}_zI_y)({}_yD_y) $$
$$ {}_yD_z = \qquad ({}_yD_y)({}_yI_z) $$