Homework 17 MATH 304 Section 3

Assigned: Wednesday, October 29.
Potentially Collected: Wednesday, November 5.

1. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane defined by $x + 2y + z = 0$. Hint: Think of the equation as a system of linear equations.

2. Let $\mathbb{F}$ be the set of all real-valued functions. That is, $f \in F$ is such that $f : \mathbb{R} \to \mathbb{R}$.
   - Show that $\mathbb{F}$ is a vector space with scalars $\mathbb{R}$.
   - Find a basis for the subspace spanned by $\{\overset{\vec{v}_1}{\sin(x)}, \overset{\vec{v}_2}{\sin(2x)}, \overset{\vec{v}_3}{\sin(x)\cos(x)}\} = S$

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① $x+2y+z=0$ is an equation where $[1 \ 2 \ 1 \,\vdots\, 0]$

Elements $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ are "on" the plane (well, the standard vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ has tail on the origin and head on the plane)

when $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in Ker(F)$ where

F is the matrix $[1 \ 2 \ 1]$

notice that y, z are free and solutions are of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y+z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The plane (a.k.a. $Ker(F)$) is 2 dimensional and has a basis $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

S LI? $\vec{0} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = a_1\sin(x) + a_2\sin(2x) + a_3\sin(x)\cos(x)$

[notice that $\vec{v}_2 = \sin(2x) = 2\sin(x)\cos(x) = 2\vec{v}_3$ LD!

so $Span(\vec{v}_1, \vec{v}_2, \vec{v}_3) = span(\vec{v}_1, \vec{v}_3)$ (span preservation)

If $a_1\vec{v}_1 = \vec{v}_3 \Rightarrow a_1\sin(x) - \sin(x)\cos(x) = 0$

$\Rightarrow \underbrace{\sin(x)}_{\text{not zero!}}\underbrace{(a_1 - \cos(x))}_{\text{not constant!}} = 0$

So $\{\vec{v}_1, \vec{v}_3\}$ is LI and spans $Span(S)$, hence it is a basis.

② $\mathbb{F}$ consists of functions with domain and codomain $\mathbb{R}$.

vector addition $f+g : \mathbb{R} \to \mathbb{R}$ where $(f+g)(x) = f(x)+g(x)$

scalar multiplication $cf : \mathbb{R} \to \mathbb{R}$ where $(cf)(x) = c\,f(x)$

Additive

for any f, g, h

commutative $(f+g)(x) = f(x)+g(x)$

codomain is commutative and associative $\to = g(x)+f(x) = (g+f)(x)$

associative $((f+g)+h)(x) = (f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$
$= (f+(g+h))(x)$

Identity

$f(x)=0$ acts like $\vec{0}$.

Inverse

$f(x)+(-f)(x) = 0$ for any f

Multiplication

Identity $1f(x) = f(x)$ for any f

Associative $(cd)f(x) = c(df)(x)$ for any f and any $c,d \in \mathbb{R}$

Distribution

for any f,g any $c,d \in \mathbb{R}$

$(c(f+g))(x) = c(f(x)+g(x)) = (cf)(x)+(cg)(x)$

$((c+d)f)(x) = (c+d)f(x) = (cf)(x)+(df)(x)$

Therefore, $\mathbb{F}$ is a vector space with scalars from $\mathbb{R}$.