Homework 16 MATH 304 Section 3 Solution

Assigned: Monday, October 27.
Potentially Collected: Monday, November 3.

1. For each of the following (i) find a basis $\mathcal{B}$ for the subspace of V spanned by $\mathcal{X}$ (ii) find a basis $\mathcal{C}$ for V which contains $\mathcal{B}$.

   (a) V is $\mathbb{P}_2$ and $\mathcal{X} = \{4x^2 - 3x + 7, \quad x^2 + 9x - 2, \quad 7x^2 - 11x + 6\}$

   (b) V is $\mathbb{P}_3$ and $\mathcal{X} = \{x^3 - 1, \quad -2x^3 + x^2 - x + 1, \quad 6x^3 - x^2 + 2x - 1, \quad 3x^2 - x + 1\}$

   (c) V is $\mathbb{P}_3$ and

$$\mathcal{X} = \{x^3 - 3x + 2, \; x^2 + 2x - 3, \; -3x^3 - 4x^2 + x + 6, \; x^3 - 3x^2 - 8x + 7, \; 2x^3 + x^2 - 6x + 9\}$$

(a) using the ordered basis $\mathcal{Y} = (x^2, x, 1)$ for $\mathbb{P}_2$ we can convert $\mathbb{P}_2$ elements to $\mathbb{R}^3$ elements through $K_{\mathcal{Y}}$

$$K_{\mathcal{Y}}(4x^2-3x+7) = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}, \quad K_{\mathcal{Y}}(x^2+9x-2) = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$$

$$K_{\mathcal{Y}}(7x^2-11x+6) = \begin{bmatrix} 7 \\ -11 \\ 6 \end{bmatrix}$$

(i) $\begin{bmatrix} 4 & 1 & 7 \\ -3 & 9 & -11 \\ 7 & -2 & 6 \end{bmatrix}$ in RREF is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so $\mathcal{X}$ is LI and is a basis for Span($\mathcal{X}$).

(ii) As $\mathbb{P}_2$ has dimension 3, $\mathbb{P}_2 = \text{Span}(\mathcal{X})$.
Let $\mathcal{B} = \mathcal{X}$ and $\mathcal{C} = \mathcal{X}$.

(b) Using the ordered basis $\mathcal{Y} = (x^3, x^2, x, 1)$ for $\mathbb{P}_3$

$$K_{\mathcal{Y}}(x^3-1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad K_{\mathcal{Y}}(-2x^3+x^2-x+1) = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$K_{\mathcal{Y}}(6x^3-x^2+2x-1) = \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad K_{\mathcal{Y}}(3x^2-x+1) = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

(i) in RREF $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Therefore $\mathcal{X}$ is LI and $\mathcal{X}$ is a basis for Span$(\mathcal{X})$. Let $\mathcal{B} = \mathcal{X}$.

(ii) As $\dim(\mathbb{P}_3) = 4$ and $\mathbb{P}_3 = \text{Span}(\mathcal{X})$, we can take $\mathcal{C}$ to be $\mathcal{X}$.

(c) As in (b) we use $\mathcal{Y}$

$K_\mathcal{Y}(x^3 - 3x + 2) = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}$, $K_\mathcal{Y}(x^2 + 2x - 3) = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}$

$K_\mathcal{Y}(-3x^3 - 4x^2 + x + 6) = \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}$, $K_\mathcal{Y}(x^3 - 3x^2 - 8x + 7) = \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}$

$K_\mathcal{Y}(2x^3 + x^2 - 6x + 9) = \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$ in RREF $\begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(i) $\mathcal{B} = \{x^3 - 3x + 2,\ x^2 + 2x - 3,\ x^3 - 3x^2 - 8x + 7\}$ is a basis for Span$(\mathcal{X})$.

(ii) $\mathbb{P}_4$ is 4-dimensional so we must independence extend $\mathcal{B}$ once to find a basis for $\mathbb{P}_4$.

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -3 & b \\ -3 & 2 & -8 & c \\ 2 & -3 & 7 & d \end{array}\right]$ $\begin{array}{l} r_3 = R_3 + 3R_1 \\ r_4 = R_4 - 2R_1 \end{array}$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -3 & b \\ 0 & 2 & -5 & c+3a \\ 0 & -3 & 5 & d-2a \end{array}\right]$ $\begin{array}{l} r_3 = R_3 - 2R_2 \\ r_4 = R_4 + 3R_2 \end{array}$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -3 & b \\ 0 & 0 & 1 & c+3a-2b \\ 0 & 0 & -4 & d-2a+3b \end{array}\right]$

$r_4 = R_4 + 4R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -3 & b \\ 0 & 0 & 1 & c+3a-2b \\ 0 & 0 & 0 & d+4c-5b+10a \end{array}\right]$

so $\mathcal{C} = \mathcal{B} \cup \{1\}$ is a basis for $\mathbb{P}_4$

choose any vector where $d + 4c - 5b + 10a \neq 0$