Homework 15 MATH 304 Section 3

Solution

Assigned: Friday, October 24.
Potentially Collected: Friday, October 31.

1. For each of the following (i) find a basis $\mathcal{B}$ for the subspace of V spanned by $\mathcal{X}$ (ii) find a basis $\mathcal{C}$ for V which contains $\mathcal{B}$.

   (a) V is $\mathbb{R}^2$ and $\mathcal{X} = \left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$

   (b) V is $\mathbb{R}^3$ and $\mathcal{X} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix} \right\}$

   (c) V is $\mathbb{R}^3$ and $\mathcal{X} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \right\}$

(a) $\mathcal{X}$ in RREF is $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(i) Therefore, $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$ is a basis for Span$(\mathcal{X})$.

(ii) let $\mathcal{C} = \mathcal{B}$ as $\mathcal{B}$ is also a basis for $\mathbb{R}^2 = \text{Span}(\mathcal{X})$.

(b) $\mathcal{X}$ in RREF is $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. (i) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for Span$(\mathcal{X})$.

(ii) we use independence extension to find a basis $\mathcal{C}$ for $\mathbb{R}^3$.

$$\left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & 1 & y \\ 2 & 1 & z \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & 1 & y \\ 0 & -5 & z-2x \end{array}\right] \xrightarrow{R_3 = R_3 + 5R_2} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & 1 & y \\ 0 & 0 & z-2x+5y \end{array}\right]$$

to be out of the span$(\mathcal{X})$, a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ must satisfy $0 \neq z - 2x + 5y$

so the set

$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is LI be independence extension and it spans $\mathbb{R}^3$. $\mathcal{C}$ is a basis for $\mathbb{R}^3$ containing $\mathcal{B}$.

(a) X in RREF is $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ so

(i) the set $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \right\}$ is LI and

(ii) $\operatorname{span}(X) = \operatorname{span}(\mathcal{B})$ (by span preservation)

Let

$\mathcal{C} = \mathcal{B}$, as $\mathcal{B}$ is a basis for $\mathbb{R}^3$.