Homework 14 MATH 304 Section 3

Solution

Assigned: Wednesday, October 22.
Potentially Collected: Wednesday, October 29.

1. Given a vector space V, a basis X, and a vector $\vec{w}$ calculate $K_X(\vec{w})$.

   (a) $V = \mathbb{R}^3$, $X = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, and $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

   (b) $V = \mathbb{R}^4$, $X = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$, and $\vec{w} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$.

   (c) $V = \mathbb{P}_2$, $X = \{x^2+1, x+1, x^2+x\}$, and $\vec{w} = 3x^2 - x - 2$.

   (d) $V = \mathbb{P}_3$, $X = \{x^3+x^2+x+1, x^3+2x^2+x+3, 2x^3+x^2+3x+2, x^3+x^2+2x+2\}$, and $\vec{w} = x^3 - x$.

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a) $K_X(\vec{w})$ is the solution to $\left[\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 2 \end{array}\right]$, in RREF $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -5/2 \end{array}\right]$

$K_X(\vec{w}) = \frac{1}{2}\begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix}$

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b) $K_X(\vec{w})$ is the solution to $\left[\begin{array}{cccc|c} 0 & -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 \end{array}\right]$ in RREF $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 16 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & -6 \end{array}\right]$

$K_X(\vec{w}) = \begin{bmatrix} 16 \\ -4 \\ 10 \\ -6 \end{bmatrix}$

(c) $K_x(\vec{w})$ is the solution to

$$3x^2-x-2=a(x^2+1)+b(x+1)+c(x^2+x)$$
$$=(a+c)x^2+(b+c)x+(a+b)$$

$$\Rightarrow \begin{array}{l} a+c=3 \\ b+c=-1 \\ a+b=-2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$K_x(3x^2-x-2)=\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

(d) $K_x(\vec{w})$ is the solution to

$$x^3-x=a(x^3+x^2+x+1)+b(x^3+2x^2+x+3)$$
$$+c(2x^3+x^2+3x+2)+d(x^3+x^2+2x+2)$$
$$=(a+b+2c+d)x^3+(a+2b+c+d)x^2$$
$$+(a+b+3c+2d)x+(a+3b+2c+2d)$$

$$\Rightarrow \begin{array}{l} a+b+2c+d=1 \\ a+2b+c+d=0 \\ a+b+3c+2d=-1 \\ a+3b+2c+2d=0 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 & -1 \\ 1 & 3 & 2 & 2 & 0 \end{array}\right] \text{ in RREF}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]$$

$$K_x(x^3-x)=\begin{bmatrix} -2 \\ 2 \\ 3 \\ -5 \end{bmatrix}$$