Homework 27 MATH 304 Section 3

Assigned: Monday, December 8.
Potentially Collected: Friday, December 12.

1. Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \right\}$$

2. Find an orthonormal basis for the null space of each of the following matrices

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{\text{RREF}} A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \\ 1 & 2 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \end{bmatrix}$$

3. Find an orthonormal basis for each of the following subspaces.

(a) $\left\{ \begin{bmatrix} a \\ a+b \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

(b) $\left\{ \begin{bmatrix} a \\ a+b \\ c \\ b+c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

(c) $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a+b+c=0 \right\}$

(d) $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a-b-2c+d=0 \right\}$

$\text{nul}(A) = \text{span}\left(\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}\right)$

$\left\{\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}\right\}$ is orthogonal

length $\sqrt{(-4)^2+5^2+1^2} = \sqrt{42}$

$\left\{\begin{bmatrix} -4/\sqrt{42} \\ 5/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}\right\}$ is orthonormal

$\text{nul}(B) = \text{span}\left(\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}\right)$

$\left\{\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}\right\}$ is orthogonal

length $\sqrt{(-3)^2+4^2+1^2} = \sqrt{26}$

$\left\{\begin{bmatrix} -3/\sqrt{26} \\ 4/\sqrt{26} \\ 1/\sqrt{26} \end{bmatrix}\right\}$ is orthonormal

(a) $\left\{ \begin{bmatrix} a \\ a+b \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$

Gram-Schmit Process

(1) $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (2) $\vec{p}_2 = \text{proj}_{\vec{v}_1}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$

orthogonal basis

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \right\}$

$\|\vec{v}_1\| = \sqrt{2}$

$\|\vec{v}_2\| = \sqrt{1/4 + 1/4 + 1} = \sqrt{3/2}$

orthonormal basis

$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2\sqrt{3/2} \\ 1/2\sqrt{3/2} \\ 1/\sqrt{3/2} \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} a \\ a+b \\ c \\ b+c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$

Gram-Schmit Process

(1) $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (2) $\vec{p}_2 = \text{proj}_{\vec{v}_1}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right) = \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$

(3) $\vec{p}_3 = \text{Proj}_{\vec{v}_1}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) + \text{proj}_{\vec{v}_2}\left(\begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}\right) = \left(\frac{0}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \left(\frac{1}{3/2}\right)\begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/3 \\ 0 \\ 2/3 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/3 \\ 1/3 \\ 0 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \\ 1/3 \end{bmatrix}$

Orthogonal Basis

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \\ 1/3 \end{bmatrix} \right\}$

$\|\vec{v}_1\| = \sqrt{2}$

$\|\vec{v}_2\| = \sqrt{3/2}$

$\|\vec{v}_3\| = \sqrt{4/3}$

Othonormal Basis

$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2\sqrt{3/2} \\ 1/2\sqrt{3/2} \\ 1/\sqrt{3/2} \\ 1/3\sqrt{3/2} \end{bmatrix}, \begin{bmatrix} 1/3\sqrt{4/3} \\ -1/3\sqrt{4/3} \\ 1/\sqrt{4/3} \\ 1/3\sqrt{4/3} \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a+b+c=0 \right\} = \underbrace{nul\left([1 \; 1 \; 1] \right)}_{\text{b,c are free}} = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b-c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Gram-Schmit Process

① $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ ② $\vec{P}_2 = proj_{\vec{v}_1}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) = \left(\frac{1}{2} \right) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Orthogonal Basis

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$

$\|\vec{v}_1\| = \sqrt{2}$

$\|\vec{v}_2\| = \sqrt{3/2}$

Othonormal Basis

$$\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{3/2} \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a-b-2c+d=0 \right\} = nul\left([1 \; -1 \; -2 \; d] \right) = Span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b+2c-d \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Gram-Schmit Process

① $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ② $\vec{P}_2 = proj_{\vec{v}_1}\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) = \left(\frac{2}{2} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

③ $\vec{P}_3 = proj_{\vec{v}_1}\left(\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) + proj_{\vec{v}_2}\left(\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = \left(\frac{-1}{2} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \left(\frac{-1}{3} \right) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5/6 \\ -1/6 \\ -1/3 \\ 0 \end{bmatrix}$

$$\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -5/6 \\ -1/6 \\ -1/3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/6 \\ 1/3 \\ 1 \end{bmatrix}$$

$\|\vec{v}_1\| = \sqrt{2}$

$\|\vec{v}_2\| = \sqrt{3}$

$\|\vec{v}_3\| = \sqrt{7/6}$

Orthogonal Basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/6 \\ 1/6 \\ 1/3 \\ 1 \end{bmatrix} \right\}$$

Orthonormal Basis

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{42} \\ 1/\sqrt{42} \\ 2/\sqrt{42} \\ \sqrt{6/7} \end{bmatrix} \right\}$$