11/6/2014 Version A

Math 304: Linear Algebra — Midterm Exam 2

Solution

Problem	1	2	3	4	5	6	Total
Full Score	20	20	20	20	20	20	120
Your Score							

- Read all problems before beginning and try to work from easiest to hardest.
- In order to get credit, you must show all of your work.
- **NO** calculators of any kind! **NO** cell phones!
- Check to make sure that your exam has six (6) pages and six (6) questions.

1. **Clearly** circle "True" or "False" for each of the following problems. Circle "True" only if the statement is always true. No explanation necessary.

TRUE FALSE (a) Let $X = \{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}\}$, where $\vec{v_i} \in \mathbb{R}^4$. Then X is a linearly dependent set.

TRUE FALSE (b) Any set of vectors in $\mathbb{R}^n$ consisting of a single vector is a linearly independent set. $\{\vec{0}\}$ is LD

TRUE FALSE (c) If A is a invertible $n \times n$ matrix, then Nul(A)= $\{\vec{0}\}$.

TRUE FALSE (d) If A is a singular $n \times n$ matrix, then Col(A)= $\mathbb{R}^n$. $\neq$

TRUE FALSE (e) Subsets of linearly independent sets are necessarily linearly independent.

TRUE FALSE (f) A vector space has a finite spanning set if and only if it has a finite basis.

TRUE FALSE (g) Let V and W be vector spaces with $\dim(V) = n$ and $\dim(W) = m$, respectively. If $L : V \to W$ is a linear transformation which is onto, then $n \geq m$ and $\dim(\ker(L))=n-m$.

TRUE FALSE (h) Let V and W be vector spaces and F an isomorphism from V to W. If S is a linearly dependent set in V, then $F(S)$ is a linearly dependent set in W.

TRUE FALSE (i) There exists an isomorphism from $\mathbb{P}_6$ to $\mathbb{R}^6$.

TRUE FALSE (j) If F is a spanning set for a vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way. replace with basis

$$\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

2. Let $X = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right)$ be a collection of vectors in $\mathbb{R}^4$.

a) Show that X is linearly dependent by finding a linear dependence relation for X.

in RREF we have $\begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. X is LD as the rank is 2 which does not equal the number of elements in X.

$\vec{v}_3 = 5/2\,\vec{v}_1 - 1/2\,\vec{v}_2$

$\Rightarrow \vec{0} = 5/2\,\vec{v}_1 - 1/2\,\vec{v}_2 - \vec{v}_3$

b) Name a vector that can be removed from X so that the remaining two vectors have the same span as X. **Explain!**

As all three vectors have nonzero coefficients in the linear dependence,

ANY vector may be removed. (span preservation)

c) For every real number t, let $\vec{v}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ t \end{bmatrix}$.

Determine all values for t such that $\vec{v}(t)$ belongs to Span(X). For each such value of t, express $\vec{v}(t)$ as a specific numeric linear combination of vectors from X. **Show your work!**

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 4 & 3 & 0 \\ 1 & 3 & 1 & -1 \\ 1 & 5 & 0 & t \end{array}\right] \begin{array}{l} r_2 = R_2 - 2R_1 \\ r_3 = R_3 - R_1 \\ r_4 = R_4 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & -1 & -2 \\ 0 & 2 & -1 & -2 \\ 0 & 4 & -2 & t-1 \end{array}\right] \begin{array}{l} r_3 = R_3 - R_2 \\ r_4 = R_4 - 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t-3 \end{array}\right]$$

$v(t) \in \text{span}(X)$ if and only if $t - 3 = 0$

$\Rightarrow \underline{t = 3}$

$v(3) = 2\vec{v}_1 - \vec{v}_2$

3. Consider the following vectors in $\mathbb{R}^3$:

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix}.$$

let $\mathcal{X} = \{\vec{u}, \vec{v}, \vec{w}\}$

(a) Find all $t \in \mathbb{R}$ such that $\vec{u}, \vec{v}, \vec{w}$ form a basis for $\mathbb{R}^3$. $\{t \mid t \neq -1\}$

$$\begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & t \\ 1 & 3 & 1 \end{bmatrix} \begin{matrix} r_2 = R_2 - R_1 \\ r_3 = R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & t+2 \\ 0 & 3 & 3 \end{bmatrix} r_3 = R_3 - 3R_2 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & t+2 \\ 0 & 0 & -3t-3 \end{bmatrix}$$

$\mathcal{X}$ is a basis for $\mathbb{R}^3$ if and only if the matrix for $\mathcal{X}$ is invertible if and only if $-3t-3 \neq 0$ $\Rightarrow t \neq -1$

(b) It is given that for $t = 0$ the vectors $\vec{u}, \vec{v}, \vec{w}$ form a basis for $\mathbb{R}^3$. Let $B_1 = (\vec{u}, \vec{v}, \vec{w})$ and $B_2 = (\vec{v}, \vec{u}, \vec{w})$ be ordered bases of $\mathbb{R}^3$.

(i) Find $K_{B_1}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$ (recall that K_Y is the coordinate transformation for the basis Y).

$$K_{B_1}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ where } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = a\vec{u} + b\vec{v} + c\vec{w}$$

Solve $\left[\vec{u}\ \vec{v}\ \vec{w} \,\middle|\, \begin{matrix} 1 \\ 0 \\ 1 \end{matrix}\right]$

in RREF $\left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \,\middle|\, \begin{matrix} -1 \\ 1 \\ -1 \end{matrix}\right]$

So

$$K_{B_1}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

(ii) Find a vector $\vec{b} \in \mathbb{R}^3$ such that $K_{B_2}(\vec{b}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$$\vec{b} = 1\vec{v} + 0\vec{u} + 1\vec{w}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

$$K_{B_2}\left(\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Version B

$$K_{B_1}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$K_{B_2}\left(\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

4. Let $M = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 \\ 1 & 2 & -1 & 2 & 0 \\ -2 & -1 & 5 & -7 & 4 \end{bmatrix}$. in RREF $\begin{bmatrix} 1 & 0 & -3 & 4 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(a) Complete the following:

Nul(M) is a subspace of $\mathbb{R}^g$ where $g =$ 5.

Col(M) is a subspace of $\mathbb{R}^h$ where $h =$ 3.

(b) Determine Nul(M) and a basis for Nul(M).

$Nul(M) = \{\vec{x} \in \mathbb{R}^5 \mid M\vec{x} = \vec{0}\}$

the solution to $M\vec{x} = \vec{0}$ can be expressed

$c_1 - 3c_3 + 4c_4 = 0$
$c_2 + c_3 - c_4 = 0$
$c_5 = 0$

so $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = c_3 \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

A basis is $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(c) Find a basis for Col(M).

A basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right\}$

Version B

A basis for Nul(M) is

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

A basis for Col(M) is

$\left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -8 \end{bmatrix} \right\}$

(d) Find a basis for Row(M).

A basis is $\left\{ \begin{matrix} [0\ 1\ 1\ -1\ 1], \\ [1\ 2\ -1\ 2\ 0], \\ [-2\ -1\ 5\ -7\ 4] \end{matrix} \right\}$

or $\left\{ \begin{matrix} [1\ 0\ -3\ 4\ 0], \\ [0\ 1\ 1\ -1\ 0], \\ [0\ 0\ 0\ 0\ 1] \end{matrix} \right\}$

For row(M) is

$\left\{ \begin{matrix} [0\ 1\ -1\ 1\ 1], \\ [1\ 2\ 2\ -1\ 0], \\ [-3\ -2\ -8\ 5\ 3] \end{matrix} \right\}$

or $\left\{ \begin{matrix} [1\ 0\ 0\ 1\ 0], \\ [0\ 1\ 0\ 0\ 1/2], \\ [0\ 0\ 1\ -1\ -1/2] \end{matrix} \right\}$

5. Let $\mathbb{P}_3$ be the vector space of polynomials with real coefficients of degree at most 3. Consider $\mathcal{X} = \{1 - 2x + x^3 \ , \ -2 + 2x + x^2 + 2x^3 \ , \ 1 + x - 2x^2\}$.

(a) Show that $\mathcal{X}$ is a linearly independent set. let $\mathcal{Y} = (1, x, x^2, x^3)$

$$K_{\mathcal{Y}}(\mathcal{X}) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\}$$

the matrix for $K_{\mathcal{Y}}(\mathcal{X})$ has RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so $K_{\mathcal{Y}}(\mathcal{X})$ is LI. (rank = #elements)

As $K_{\mathcal{Y}}$ is an isomorphism, $\mathcal{X}$ is also LI.

(b) Find a basis for Span($\mathcal{X}$)). What is the dimension of Span($\mathcal{X}$))?

$\mathcal{X}$ is a spanning set for Span($\mathcal{X}$) and it is LI by (a).

$\mathcal{X}$ is a basis for Span($\mathcal{X}$).

$\dim(\text{Span}(\mathcal{X})) = 3$

(c) Find a basis of $\mathbb{P}_3$ which contains the set $\mathcal{X}$. Explain why the set you find is a basis.

Independence Extension: Find $\vec{v} \notin \text{Span}(\mathcal{X})$.

$\mathcal{X} \cup \{\vec{v}\}$ is LI and span $(\mathbb{P}_3)$, so $\mathcal{X} \cup \{\vec{v}\}$ is a basis for $\mathbb{P}_3$ containing $\mathcal{X}$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & a \\ -2 & 2 & 1 & b \\ 0 & 1 & -2 & c \\ 1 & 2 & 0 & d \end{array}\right]$$ must be inconsistent.

A possible solution is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, so

$\mathcal{X} \cup \{x^3\}$ will soffice.

6. Let $\mathbb{P}_3$ be the vector space consisting of polynomials with degree less than or equal to 3.

(a) What is $\dim(\mathbb{P}_3)$? 4

(b) Consider the function $F : \mathbb{P}_3 \longrightarrow \mathbb{R}$ defined by $F(p(x)) = p(0) = a_0$ where $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Show that F is a linear transformation.

i) $F: \mathbb{P}_3 \to \mathbb{R}$ takes a vector space to another vector space.

ii) for $p(x), q(x) \in \mathbb{P}_3$ $\quad F(p(x)+q(x)) = F((p+q)(x)) = (p+q)(0)$
$= p(0) + q(0) = F(p(x)) + F(q(x))$

iii) for $c \in \mathbb{R}$, $p(x) \in \mathbb{P}_3$ $\quad F(cp(x)) = F((cp)(x)) = cp(0) = cF(p(x))$

(c) Show that F is onto (surjective).

for any $c \in \mathbb{R}$, the polynomial $p(x) = c$ is such that
$F(p(x)) = p(0) = c$.

Alternatively, the matrix for F is $[F(1)\ F(x)\ F(x^2)\ F(x^3)]$
$= [1\ 0\ 0\ 0]$

$\uparrow$ rank = # rows

(d) What is $\dim(\ker(F))$?

$\dim(\ker(F)) = \dim(\mathbb{P}_3) - \dim(\mathrm{Im}(F))$
$= 4 - \dim(\mathbb{R}) = 4 - 1 = 3$

(e) Is $S = (x, x + x^2, x + x^2 + x^3)$ a basis for $\ker(F)$? Explain.

$\mathcal{Y} = (1, x, x^2, x^3)$

$F(x) = 0$
$F(x+x^2) = 0$
$F(x+x^2+x^3) = 0$
so
$S \subseteq \ker(F)$

S is LI because $K_{\mathcal{Y}}(S) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is LI.

S spans $\ker(F)$ as it is LI and $\dim(\ker(F)) = 3$,
so S is a basis for $\ker(F)$.