10/2/2014 Version B

# Math 304: Linear Algebra Midterm Exam 1

Solution

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Full Score | 20 | 20 | 20 | 20 | 20 | 20 | 120 |
| Your Score | | | | | | | |

- Read all problems before beginning and try to work from easiest to hardest.
- In order to get credit, you must show all of your work.
- **NO** calculators of any kind! **NO** cell phones!
- Check to make sure that your exam has six (6) pages and six (6) questions.

1. **Clearly** circle "True" or "False" for each of the following problems. Circle "True" only if the statement is always true. No explanation necessary.

TRUE FALSE (a) The matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is in diagonal form.

TRUE FALSE (b) The rank of an $11 \times 7$ matrix is greater or equal to 7.

TRUE FALSE (c) Suppose that $A$ is a $5 \times 3$ matrix. Then $A\vec{x} = \vec{0}$ has infinitely many solutions.

TRUE FALSE (d) Let $A$ be an $m \times n$ matrix with $m > n$. Then any row echelon form contains at least $m - n$ zero rows.

TRUE FALSE (e) An $m \times n$ matrix has $m$ rows and $n$ columns.

TRUE FALSE (f) Every elementary matrix is nonsingular.

TRUE FALSE (g) Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be different solutions of the nonhomogeneous system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Then $\vec{\mathbf{u}} - \vec{\mathbf{v}}$ is a nontrivial solution of the associated homogeneous system.

TRUE FALSE (h) The matrix $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ is in reduced row echelon form.

TRUE FALSE (i) Let $A, B$ be $n \times n$ matrices and $rank(A) = n$. Then the matrix equation $AX = B$ is always solvable.

TRUE FALSE (j) $(AB^T)^T$ is always equal to $A^T B$ for all matrices $A$ and $B$ such that $AB^T$ is defined.

2. Suppose that

$$M = \left[\begin{array}{ccccc|c} 0 & 1 & -1 & -3 & 4 & 13 \\ 0 & 3 & -3 & -9 & 15 & 45 \\ 0 & -1 & 4 & 9 & -4 & -19 \end{array}\right]$$

$r_2 = R_2 - 3R_1$
$r_3 = R_3 + R_1$

is the augmented matrix of a system of linear equations in the variables $x_1, x_2, x_3, x_4, x_5$.

a) Bring the matrix $M$ into reduced row echelon form, indicating all elementary row operations.

$$\left[\begin{array}{ccccc|c} 0 & 1 & -1 & -3 & 4 & 13 \\ 0 & 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 & 0 & -6 \end{array}\right] \quad r_2 = \tfrac{1}{3}R_3 \quad r_3 = \tfrac{1}{3}R_2 \quad \left[\begin{array}{ccccc|c} 0 & 1 & -1 & -3 & 4 & 13 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad r_1 = R_1 - 4R_3$$

$$\left[\begin{array}{ccccc|c} 0 & 1 & -1 & -3 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad r_1 = R_1 + R_2 \quad \left[\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

b) Which variables are the basic variables?

$x_2, x_3, x_5$

c) Which variables are the free variables?

$x_1, x_4$

d) What is the rank of $M$?

3

e) List the columns of $M$ which are pivot columns.

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 15 \\ -4 \end{bmatrix}$$

f) If the system is consistent, write its solution in parametric form.

$x_1$ anything

$x_2 = 3 + 1x_4$

$x_3 = -2 - 2x_4$

$x_4$ anything

$x_5 = 2$

3. Given a matrix

$$M = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & a & b \\ 0 & 1 & 2 & 3 & c & d \\ 0 & 0 & 0 & 0 & e & f \end{array}\right]$$

representing the augmented matrix of a system of equations in reduced row echelon form. Compute the following by filling in the blanks.

(a) For $a =$ anything, $b =$ 0, $c =$ anything, $d =$ 0, $e =$ 0, $f =$ 1, the matrix $M$ represents the $\underbrace{\text{reduced row echelon}}_{f=1,\ b=0,\ d=0}$ form of an $\underbrace{\text{inconsistent}}_{\to e=0,\ f\neq 0}$ system of equations.

$f=1,\ b=0,\ d=0$ $\quad\to e=0,\ f\neq 0$

The pivots are located at $m_{11}$, $m_{22}$, $m_{36}$. (Give your answer in the form $m_{ij}$.)

The rank of the coefficient matrix is 2.

The rank of the augmented matrix is 3.

(b) For $a =$ 0, $b =$ anything, $c =$ 0, $d =$ anything, $e =$ 1, $f = 2$, the matrix $M$ is the augmented matrix of a consistent nonhomogeneous system in reduced row echelon form.

$e\neq 0$ $\quad\to b, d, f$ not all zero $\quad\to e=1, a=0, c=0$

The pivots are located at $m_{11}$, $m_{22}$, $m_{35}$.

The rank of the augmented matrix is 3.

The rank of the coefficient matrix is 3.

(c) For $a = 1$, $b =$ 0, $c = 1$, $d =$ 0, $e =$ 0, $f =$ 0, the matrix $M$ is the augmented matrix of a homogeneous system of rank 2 in reduced row echelon form.

$\to b, d, f = 0$ $\quad\to e=0$

The complete solution in parameterized form is

$x_1 =$ ______, $x_2 =$ ______, $x_3 =$ ______, $x_4 =$ ______, $x_5 =$ ______.

$x_1 = -x_3 - x_4 - x_5$ $\qquad x_3, x_4, x_5$ anything

$x_2 = 2x_3 - 3x_4 - x_5$

4. Consider the matrices $A = \begin{bmatrix} 2 & -1 & -4 \\ -1 & \frac{1}{2} & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}$. For each of the following operations, either do the indicated calculations or explain why it is not defined.

(i) $A + B$

Undefined

(ii) $A \cdot B$

$\underset{2\times 3}{A} \quad \underset{2\times 2}{B}$ undefined

(iii) $B \cdot A$

$$= \begin{bmatrix} [1\ 2]A \\ [\frac{1}{2}\ 1]A \end{bmatrix} = \begin{bmatrix} 1[2\ -1\ -4] + 2[-1\ \frac{1}{2}\ 2] \\ \frac{1}{2}[2\ -1\ -4] + 1[-1\ \frac{1}{2}\ 2] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(iv) $(B \cdot A)^2$

$(2\times 3)^2$ is undefined

(v) $2A^T + 8A^T \cdot B$

$$A^TB = \begin{bmatrix} [2\ -1]B \\ [-1\ \frac{1}{2}]B \\ [-4\ 2]B \end{bmatrix} = \begin{bmatrix} 2[1\ 2] - [\frac{1}{2}\ 1] \\ -1[1\ 2] + \frac{1}{2}[\frac{1}{2}\ 1] \\ -4[1\ 2] + 2[\frac{1}{2}\ 1] \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 3 \\ -\frac{3}{4} & -\frac{3}{2} \\ -3 & -6 \end{bmatrix}$$

$$2A^T + 8A^TB = \begin{bmatrix} 4 & -2 \\ -2 & 1 \\ -8 & 4 \end{bmatrix} + \begin{bmatrix} 12 & 24 \\ -6 & -12 \\ -24 & -48 \end{bmatrix} = \begin{bmatrix} 16 & 22 \\ -8 & -11 \\ -32 & -44 \end{bmatrix}$$

(vi) $A^T \cdot B^T$.

$$A^TB^T = \begin{bmatrix} [2\ -1]B^T \\ [-1\ \frac{1}{2}]B^T \\ [-4\ 2]B^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(vii) How are the matrices $B \cdot A$ and $A^T \cdot B^T$ related? Justify your answer.

$$A^TB^T = (BA)^T$$

5. Let $A$ be the vector space consisting of column vectors of length 4 and let $B$ be the vector space of column vectors of length 3. Consider the function $f : A \longrightarrow B$ given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 4x_2 + 2x_3 + 2x_4 \\ x_1 + 2x_2 + 2x_3 + 2x_4 \\ x_1 + 2x_2 + x_3 + 2x_4 \end{bmatrix}$$

$f : \mathbb{R}^4 \to \mathbb{R}^3$

(a) What is the domain of $f$? What is the codomain?

$\mathbb{R}^4$ $\mathbb{R}^3$

(b) Determine $f(\vec{e_i})$ for $i = 1, 2, 3, 4$ for the standard basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}\}$ of $\mathbb{R}^4$ written as column vectors.

$$f(\vec{e_1}) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},\ f(\vec{e_2}) = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix},\ f(\vec{e_3}) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},\ f(\vec{e_4}) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

(c) Using (b), write down the standard matrix $M$ such that $f(\vec{x}) = M \cdot \vec{x}$.

$$M = \begin{bmatrix} 2 & 4 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

(d) Determine the rank of $M$.

$$r_1 = \tfrac{1}{2}R_1 \quad \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix} \quad \begin{matrix} r_2 = R_2 - R_1 \\ r_3 = R_3 - R_1 \end{matrix} \quad \underbrace{\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{REF}$$

$\text{rank}(M) = 3$

(e) Is the function $f$ one-to-one? Explain. NO

$\text{rank}(M) = 3 \neq 4 = \#\text{columns}.$

(f) Is the function $f$ onto? Explain. YES

$\text{rank}(M) = 3 = \#\text{rows}.$

6. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 0 & 2 & 1 \end{bmatrix}$.

(a) Find $A^{-1}$ and check your result.

Solve $AX = I_2$

$$[A \,\vdots\, I_2] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{r_2 = R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{r_1 = R_1 - 3R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right] = [I_2 \,\vdots\, A^{-1}] \qquad A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

<u>check:</u> $AA^{-1} = A^{-1}A = I_2$

(b) Use your work from part (a) to express $A^{-1}$ and then $A$ as a product of elementary matrices.

$r_2 = R_2 - 2R_1$ corresponds to multiplication by $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$r_1 = R_1 - 3R_2$ " " " $E_2 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$$A^{-1} = E_2 E_1$$

$$A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

(c) Solve the matrix equation $AX = B$ using $A^{-1}$ from part (a).
(You must use $A^{-1}$, not any other method.)

$$AX = B \Rightarrow A^{-1}AX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

be very careful with commutativity!

$$X = A^{-1}B = \begin{bmatrix} 1 & 0 & 1 & -10 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$