Test 2 MATH 222 10/22/2014

Solutions

There are two colors (versions) of the exam, blue and white. Each problem has one or two solutions depending on whether the problems differed between the versions.

1a. $\int_0^{\frac{\sqrt{\pi}}{2}} x \tan x^2 \, dx$. (both versions)

Let $u = x^2$, $du = 2x\, dx$, $x\, dx = \frac{1}{2}\, du$.

With $x = 0$, $u = 0$, and with $x = \dfrac{\sqrt{\pi}}{2}$, $u = \frac{\pi}{4}$.

$$\begin{aligned}
\int_0^{\frac{\sqrt{\pi}}{2}} x \tan x^2 \, dx &= \int_0^{\frac{\pi}{4}} \tan u \, \frac{du}{2} \\
&= \frac{1}{2} \left[\ln|\sec u|\right] \Big|_0^{\frac{\pi}{4}} \\
&= \frac{1}{2} \left[\ln|\sec \frac{\pi}{4}| - \ln|\sec 0|\right] \\
&= \frac{1}{2} \left[\ln(\sqrt{2}) - \ln(1)\right] = \frac{1}{2}\ln(\sqrt{2}) = \frac{1}{4}\ln(2).
\end{aligned}$$

1b. $\int_0^{\pi} x \cos x \, dx$. (blue version) Let $u = x$, $dv = \cos x\, dx$, $du = dx$, $v = \sin x$.

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x - (-\cos x) + C$$

$$\int_0^{\pi} x \cos x \, dx = (x \sin x + \cos x)\Big|_0^{\pi} = (\pi \sin \pi + \cos \pi) - (0 + \cos 0) = -2.$$

1b. $\int_0^{\pi} x \sin x \, dx$. (white version)

Let $u = x$, $dv = \sin x\, dx$, $du = dx$, $v = -\cos x$.

$$\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \sin x + C$$

$$\int_0^{\pi} x \sin x \, dx = (-x \cos x + \sin x)\Big|_0^{\pi} = (-\pi \cos \pi + 0) - (0) = -\pi(-1) = \pi.$$

1c. $\int x^3 \ln x\, dx$. (blue version)

Let $du = x^3$, $v = \ln x$, $u = \dfrac{x^4}{4}$, $dv = \dfrac{1}{x}\, dx$.

$$\int x^3 \ln x\, dx = \frac{1}{4} x^4 \ln x - \int \frac{x^3}{4}\, dx = \frac{1}{4} x^4 \ln x - \frac{x^4}{16} + C.$$

1c. $\int x^4 \ln x\, dx$. (white version)

Let $du = x^4$, $v = \ln x$, $u = \dfrac{x^5}{5}$, $dv = \dfrac{1}{x}\, dx$.

$$\int x^4 \ln x\, dx = \frac{1}{5} x^5 \ln x - \int \frac{x^4}{5}\, dx = \frac{1}{5} x^5 \ln x - \frac{x^5}{25} + C.$$

1d. $\int \dfrac{x^3}{x^2+1}\, dx$. (both versions)

Dividing x^2+1 into x^3 gives $\dfrac{x^3}{x^2+1} = x - \dfrac{x}{x^2+1}$. Note $\dfrac{d}{dx}(x^2+1) = 2x$.

$$\int \frac{x^3}{x^2+1}\, dx = \int \left(x - \frac{1}{2} \frac{2x}{x^2+1} \right) dx = \frac{x^2}{2} - \frac{1}{2} \ln(x^2+1) + C.$$

2a. $\int \tan^4 x\, dx$. (both versions)

$$\int \tan^4 x\, dx = \int \tan^2 x (\sec^2 x - 1)\, dx = \int (\tan^2 x \sec^2 x - \tan^2 x)\, dx$$

$$\int \tan^2 x \sec^2 x\, dx = \int u^2\, du = \frac{u^3}{3} + C = \frac{\tan^3 x}{3} + C$$

using $u = \tan x$, and

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C,$$

$$\int \tan^4 x\, dx = \frac{\tan^3 x}{3} - (\tan x - x) + C = \frac{\tan^3 x}{3} - \tan x + x + C.$$

2b. $\int \dfrac{5x-1}{x^2-x-2}\, dx$. (both versions) $x^2 - x - 2 = (x-2)(x+1)$ and

$$\frac{5x-1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{Ax + A + Bx - 2B}{(x-2)(x+1)} = \frac{(A+B)x + (A-2B)}{(x-2)(x+1)}$$

$$5 = A + B, \qquad A = 5 - B$$

$$-1 = A - 2B = 5 - B - 2B = 5 - 3B, \qquad -6 = -3B \qquad B = 2$$

$$A = 5 - B = 5 - 2 = 3$$

$$\int \frac{5x-1}{x^2-x-2}\, dx = \int \frac{3}{x-2}\, dx + \int \frac{2}{x+1}\, dx$$

$$= 3 \ln|x-2| + 2 \ln|x+1| + C = \ln|(x-2)^3 (x+1)^2| + C.$$

2c. $\displaystyle\int \frac{dx}{\sqrt{3-x^2-2x}}\,dx$. (both versions)

$3-x^2-2x = -(x^2+2x-3) = -(x^2+2x+1-4) = -((x+1)^2-4) = 4-(x+1)^2.$

Let $x+1 = 2\sin u,\ dx = 2\cos u\,du,\ u = \sin^{-1}\left(\frac{x+1}{2}\right).$

$$\begin{aligned}\int \frac{dx}{\sqrt{3-x^2-2x}} &= \int \frac{2\cos u\,du}{\sqrt{4-4\sin^2 u}} = \int \frac{\cos u\,du}{\sqrt{\cos^2 u}} = \int du = u + C \\ &= \sin^{-1}\left(\frac{x+1}{2}\right) + C\end{aligned}$$

2d $\displaystyle\int \sin^4 x\,dx$. (blue version)

$$\begin{aligned}\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx &= \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx \\ &= \frac{1}{4}\int \left(1-2\cos(2x)+\cos^2(2x)\right)\,dx \\ &= \frac{1}{4}\int \left(1-2\cos(2x)+\frac{1+\cos(4x)}{2}\right) dx \\ &= \frac{1}{4}\int \left(\frac{3}{2}-2\cos(2x)+\frac{1}{2}\cos(4x)\right) dx \\ &= \frac{1}{4}\left(\frac{3}{2}x-\sin(2x)+\frac{1}{2}\frac{\sin(4x)}{4}\right)+C \\ &= \frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C\end{aligned}$$

2d $\displaystyle\int \cos^4 x\,dx$. (white version)

$$\begin{aligned}\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx \\ &= \frac{1}{4}\int \left(1+2\cos(2x)+\cos^2(2x)\right)\,dx \\ &= \frac{1}{4}\int \left(1+2\cos(2x)+\frac{1+\cos(4x)}{2}\right) dx \\ &= \frac{1}{4}\int \left(\frac{3}{2}+2\cos(2x)+\frac{1}{2}\cos(4x)\right) dx \\ &= \frac{1}{4}\left(\frac{3}{2}x+\sin(2x)+\frac{1}{2}\frac{\sin(4x)}{4}\right)+C \\ &= \frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C\end{aligned}$$

3(i). $\displaystyle\int_0^\infty \frac{dx}{x^2+6x+10}$. (both versions)

$$\int_0^\infty \frac{dx}{x^2+6x+10} = \int_0^\infty \frac{dx}{x^2+6x+9+1} = \int_0^\infty \frac{dx}{(x+3)^2+1}$$
$$= \lim_{a\to\infty}\int_0^a \frac{dx}{(x+3)^2+1} = \lim_{a\to\infty}\left[\tan^{-1}(a+3) - \tan^{-1}3\right]$$
$$= \frac{\pi}{2} - \tan^{-1}3$$

where we used $(x+3) = \tan u$ and $dx = \sec^2 u\,du$ so that

$$\int \frac{dx}{(x+3)^2+1} = \int \frac{sec^2 u\,du}{\sec^2 u} = \int du = u + C = \tan^{-1}(x+3) + C.$$

3(ii) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2+6x+10}\,dx$. (blue version)

$x^2+6x+10 = (x+3)^2+1 > 0$ and $1 \geq \sin^2 x \geq 0$, so

$$\frac{1}{x^2+6x+10} \geq \frac{\sin^2 x}{x^2+6x+10} \geq 0.$$

Since $\displaystyle\int_0^\infty \frac{dx}{x^2+6x+10}$ converges, $\displaystyle\int_0^\infty \frac{\sin^2 x\,dx}{x^2+6x+10}$ converges by the comparison theorem.

3(ii) $\displaystyle\int_0^\infty \frac{\cos^2 x}{x^2+6x+10}\,dx$. (white version)

Since $1 \geq \cos^2 x \geq 0$, similar to the behavior of $\sin^2 x$, the argument is similar to that of the blue version.

4. $x(t) = \dfrac{\cos^2 t}{2}$, $y(t) = \dfrac{\sin^3 t}{3}$ for $t \in \left[0, \dfrac{\pi}{2}\right]$. (blue version)

i)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin^2 t\cos t}{\cos t(-\sin t)} = -\sin t.$$
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\cos t}{\cos t(-\sin t)} = \frac{1}{\sin t} = \csc t$$

ii)

$$x(0) = \frac{1}{2}, \qquad x\left(\frac{\pi}{4}\right) = \frac{1}{4}, \qquad x\left(\frac{\pi}{2}\right) = 0,$$
$$y(0) = 0, \qquad y\left(\frac{\pi}{4}\right) = \frac{1}{6\sqrt{2}}, \qquad y\left(\frac{\pi}{2}\right) = \frac{1}{3},$$
$$\frac{dy}{dx}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \qquad \frac{d^2y}{dx^2}\left(\frac{\pi}{4}\right) = \sqrt{2}.$$

iii) from $dy/dx = -\sin t$ and $d^2y/dx^2 = \csc t$ on $[0, \pi/2]$ we know the curve is decreasing and concave up. It goes through $(0, 1/3)$ and $(1/2, 0)$.

iv)

$$A = \int_0^{1/2} y\,dx = \int_{\frac{\pi}{2}}^0 y(t)\frac{dx}{dt}\,dt = \int_{\frac{\pi}{2}}^0 \frac{\sin^3 t}{3}\cos t(-\sin t)\,dt$$
$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^4 t\cos t\,dt = \frac{1}{3}\cdot\frac{\sin^5 t}{5}\Big|_0^{\frac{\pi}{2}} = \frac{1}{15}.$$

v) This applies to both blue and white versions since switching $x(t)$ and $y(t)$ does not change the integral.

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t\sin^2 t + \sin^4 t\cos^2 t}\,dt$$
$$= \int_0^{\frac{\pi}{2}} \cos t\sin t\sqrt{1+\sin^2 t}\,dt = \int_0^1 u\sqrt{1+u^2}\,du$$

using $u = \sin t$, $du = \cos t\,dt$, $u = 0$ when $t = 0$ nd $u = 1$ when $t = \pi/2$.
Setting $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, $\theta = 0$ when $u = 0$ and $\theta = \pi/4$ when $u = 1$ gives

$$L = \int_0^{\frac{\pi}{4}} \tan\theta\sec\theta\sec^2\theta\,d\theta$$

$v = \sec\theta$, $dv = \sec\theta\tan\theta\,d\theta$, $v = 1$ when $\theta = 0$ and $v = \sqrt{2}$ when $\theta = \pi/4$ gives

$$L = \int_1^{\sqrt{2}} v^2\,dv = \frac{v^3}{3}\Big|_1^{\sqrt{2}} = \frac{1}{3}(2\sqrt{2}-1).$$

4. $x(t) = \dfrac{\sin^3 t}{3}$, $y(t) = \dfrac{\cos^2 t}{2}$ for $t \in \left[0, \dfrac{\pi}{2}\right]$. (white version: $x(t)$ and $y(t)$ are switched from the blue version)

i)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t(-\sin t)}{\sin^2 t\cos t} = \frac{-1}{\sin t} = -\csc t.$$
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\csc t\cot t}{\sin^2 t\cos t} = \frac{\cos t}{\sin t\sin^2 t\sin t\cos t} = \frac{1}{\sin^4 t} = \csc^4 t.$$

ii)

$$x(0) = 0, \qquad x\left(\frac{\pi}{4}\right) = \frac{1}{6\sqrt{2}}, \qquad x\left(\frac{\pi}{2}\right) = \frac{1}{3},$$
$$y(0) = \frac{1}{2}, \qquad y\left(\frac{\pi}{4}\right) = \frac{1}{4}, \qquad y\left(\frac{\pi}{2}\right) = 0,$$
$$\frac{dy}{dx}\left(\frac{\pi}{4}\right) = -\sqrt{2}, \qquad \frac{d^2y}{dx^2}\left(\frac{\pi}{4}\right) = 4.$$

iii) from $dy/dx = -1/\sin t$ and $d^2y/dx^2 = \csc^4 t$ on $[0, \pi/2]$ we know the curve is decreasing and concave up. It goes through $(0, 1/2)$ and $(1/3, 0)$.

iv)

$$A = \int_0^{1/3} y\,dx = \int_0^{\pi/2} y(t)\frac{dx}{dt}\,dt = \int_0^{\pi/2} \frac{\cos^2 t}{2}\sin^2 t \cos t\,dt$$

$$= \frac{1}{2}\int_0^{\pi/2}(1-\sin^2 t)(\sin^2 t)\cos t\,dt = \frac{1}{2}\int_0^1 (u^2 - u^4)\,du = \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^1 = \frac{1}{15}.$$

v) See solution for blue version.

5.

i) (both versions) $\lim\limits_{n\to\infty} = L$ means that for every $\epsilon > 0$, there is a positive integer N so that if $n > N$, then $|a_n - L| < \epsilon$.

ii) $a_n = \dfrac{3n^2 - 1}{n^2 + n + 1}$. (blue version)

a) Let $f(x) = \dfrac{3x^2 - 1}{x^2 + x + 1}$, so $a_n = f(n)$.

$$f'(x) = \frac{(x^2+x+1)(6x) - (3x^2-1)(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{6x^3 + 6x^2 + 6x - (6x^3 + 3x^2 - 2x - 1)}{(x^2+x+1)^2} = \frac{3x^2 + 8x + 1}{(x^2+x+1)^2}$$

For $x \geq 0$, both the numerator and denominator are positive so for $x \geq 0$, we have $f(x)$ is increasing and so a_n is increasing.

b)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2 - 1}{n^2 + n + 1} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \cdot \frac{3n^2 - 1}{n^2 + n + 1} = \lim_{n\to\infty} \frac{3 - \frac{1}{n^2}}{1 + \frac{1}{n} + \frac{1}{n^2}} = 3$$

ii) $a_n = \dfrac{2n^3 - 1}{n^3 + n + 1}$. (white version)

a) Let $f(x) = \dfrac{2x^3 - 1}{x^3 + x + 1}$, so $a_n = f(n)$.

$$f'(x) = \frac{(x^3+x+1)(6x) - (2x^3-1)(3x^2+1)}{(x^3+x+1)^2}$$

$$= \frac{6x^5 + 6x^3 + 6x^2 - (6x^5 + 2x^3 - 3x^2 - 1)}{(x^3+x+1)^2} = \frac{4x^3 + 9x^2 + 1}{(x^3+x+1)^2}$$

For $x \geq 0$, both the numerator and denominator are positive so for $x \geq 0$, we have $f(x)$ is increasing and so a_n is increasing.

b)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^3 - 1}{n^3 + n + 1} = \lim_{n\to\infty} \frac{\frac{1}{n^3}}{\frac{1}{n^3}} \cdot \frac{2n^3 - 1}{n^3 + n + 1} = \lim_{n\to\infty} \frac{2 - \frac{1}{n^3}}{1 + \frac{1}{n^2} + \frac{1}{n^3}} = 3$$