Math 314: Discrete Mathematics
Homework 6

Submission instructions:

a) This assignment is due Friday, March 31st at 8:00 AM.

b) The rules for submissions given on the website still apply (you still need to copy the problems down, etc.). You do not need to print off and attach this sheet (but you can).

Questions:

1. By straightforward algebra, for any k and n with $0 \leq k \leq n$,
$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$
(you just manipulate the right hand side). Using things you know about complete graphs, prove this fact without using any algebra whatsoever (explain why the left and right hand sides are equal).

2. Problem 8.5.4.

3. Are there any graphs G such that G and $\overline{G}$ are both trees? If so, what are they? Prove your solution (that there are none, or there are no others).

4. For a tree T, define $P[u, v]$ to be the subgraph that is the (unique) path between u and v. Prove that, for three distinct vertices u, v, and w in a tree, that the intersection $P[u, v] \cap P[u, w] \cap P[v, w]$ consists of a single vertex (this is called the *median* of u, v, and w).

5. Problem 9.2.3.

Optional

6. (You don't have to copy this problem down if you don't want to) The length of a path between two vertices in a graph is the number of edges in the path. The *distance* between two vertices x and y in a graph, denoted $d(x, y)$ is the minimum of the lengths of paths between the vertices (if x and y are in different components, we say $d(x, y) = \infty$). The *diameter* of a graph, $\operatorname{diam}(G)$, is the maximum distance between any two vertices (some remember this as the length of the "longest shortest path"). The *eccentricity* of a vertex, denoted $\epsilon(x)$ is the largest distance from x to any other vertex (that is, it is the maximum $d(x, y)$ for all other $y \in V(G)$. Finally, the *radius* of a graph, $\operatorname{rad}(G)$, is the minimum of all eccentricities.

a) These names are not totally arbitrary. Prove that, for any connected graph G, $\mathrm{diam}(G) \leq 2\mathrm{rad}(G)$.

b) That said, graphs are not circles. Draw a graph G that is not a complete graph (too easy) with $\mathrm{diam}(G) = \mathrm{rad}(G)$.

c) The *center* of a graph G with $\mathrm{rad}(G) = r$ is a vertex of eccentricity r. Prove that, if T is a tree, then it has either a) one center, or b) two adjacent centers (some say "the center of a tree is either a vertex or an edge").

d) Prove that, if T is a tree with one center, $\mathrm{diam}(T) = 2\mathrm{rad}(T)$, and if it has two, then $\mathrm{diam}(T) = 2\mathrm{rad}(T) - 1$.