Math 314: Discrete Mathematics
Homework 3 Solutions

1. Problem 3.8.11.: Prove that if we move straight down in Pascal's Triangle (visiting every other row), then the numbers we see are increasing.

   Proof: For any entry in Pascal's Triangle $\binom{n}{k}$ (with $n \geq k \geq 0$), the entry directly below it is $\binom{n+2}{k+1}$ (we have moved down two rows and over one diagonal). The question is then asking us to prove that

$$n \geq k \geq 0 \implies \binom{n+2}{k+1} > \binom{n}{k}.$$

   We proceed by contradiction. Assume $n \geq k \geq 0$ and

$$\binom{n+2}{k+1} \leq \binom{n}{k}.$$

   Then

$$\begin{aligned} \frac{(n+2)!}{(k+1)!((n+2)-(k+1))!} &\leq \frac{n!}{k!(n-k)!} \\ \frac{(n+2)(n+1)}{(n-k+1)!} &\leq \frac{k+1}{(n-k)!} \\ \frac{n+2}{n-k+1} &\leq \frac{k+1}{n+1} \\ &\leq 1 \qquad \text{(because } n \geq k\text{).} \end{aligned}$$

   Now, $n-k+1$ (for integers $n \geq k \geq 0$) is between 1 and $n+1$; and so will always be less than $n+2$. Then

$$\frac{n+2}{n-k+1} \leq 1$$

   is a contradiction, and we are done.

2. Prove that, if $0 \leq k \leq \frac{n-1}{2}$, then

$$\binom{n}{k} \leq \binom{n}{k+1}.$$

Further, prove equality is met if and only if $k = \frac{n-1}{2}$.

Proof: We prove the contrapositive. Assume $\binom{n}{k} > \binom{n}{k+1}$. Then

$$\begin{aligned} \frac{n!}{k!(n-k)!} &> \frac{n!}{(k+1)!(n-(k+1))!} \\ \frac{(k+1)}{(n-k)!} &> \frac{1}{(n-k-1)!} \\ k+1 &> n-k \\ 2k &> n-1 \\ k &> \frac{n-1}{2}, \end{aligned}$$

which proves the first part. For the second part:

$$\begin{aligned} \binom{n}{k} &= \binom{n}{k+1} && \text{if and only if} \\ \frac{n!}{k!(n-k)!} &= \frac{n!}{(k+1)!(n-(k+1))!} && \text{if and only if} \\ \frac{(k+1)}{(n-k)!} &= \frac{1}{(n-k-1)!} && \text{if and only if} \\ k+1 &= n-k && \text{if and only if} \\ k &= \frac{n-1}{2}. \end{aligned}$$

4. Prove that, for all $n > 0$,

$$\binom{2n}{n} < 4^n.$$

Proof: When $n = 1$ it is true because $\binom{2}{1} = 2 < 4 = 4^1$. Assume

$$\binom{2n}{n} < 4^n.$$

Then

$$\begin{aligned}
\binom{2(n+1)}{n+1} &= \frac{(2n+2)!}{(n+1)!(n+1)!} \\
&= \frac{(2n+2)(2n+1)((2n)!)}{(n+1)(n+1)n!n!} \\
&= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \cdot \frac{(2n)!}{n!n!} \\
&< \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \cdot 4^n && \text{(by induction)} \\
&= \frac{4n+2}{(n+1)} \cdot 4^n \\
&< \frac{4n+4}{(n+1)} \cdot 4^n \\
&= 4^{n+1}.
\end{aligned}$$

Clever Proof: $\binom{2n}{n}$ counts the subsets of $[2n]$ of size n, while $4^n = 2^{2n}$ is the number of *all* subsets of $[2n]$.

5. Problem 4.2.3.: Prove the following identities.

a)

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n}.$$

Proof: When $n = 1$ it is true because $F_1 = F_2 = 1$. Assume

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n}.$$

Then

$$\begin{aligned}
\sum_{i=1}^{n+1} F_{2i-1} &= \left(\sum_{i=1}^{n} F_{2i-1}\right) + F_{2n+1} \\
&= F_{2n} + F_{2n+1} && \text{(by induction)} \\
&= F_{2n+2} = F_{2(n+1)} && \text{(by definition).}
\end{aligned}$$

b)

$$\sum_{i=0}^{2n}(-1)^i F_i = F_{2n-1} - 1.$$

Proof: $(n \geq 1)$ When $n = 1$ it is true because $F_0 - F_1 + F_2 = F_1 - 1 = 0$. Assume

$$\sum_{i=0}^{2n}(-1)^i F_i = F_{2n-1} - 1.$$

Note that, because $F_{2n} + F_{2n+1} = F_{2n+2}$, $F_{2n} = -F_{2n+1} + F_{2n+2}$. Then

$$\begin{aligned}
\sum_{i=1}^{2(n+1)}(-1)^i F_i &= \left(\sum_{i=1}^{2n} F_i\right) - F_{2n+1} + F_{2n+2} \\
&= (F_{2n-1} - 1) - F_{2n+1} + F_{2n+2} && \text{(by induction)} \\
&= (F_{2n-1} - 1) + F_{2n} && \text{(by our note)} \\
&= F_{2n-1} + F_{2n} - 1 \\
&= F_{2n+1} - 1 = F_{2(n+1)-1} - 1 && \text{(by definition).}
\end{aligned}$$

c)

$$\sum_{i=0}^{n} F_i^2 = F_n \cdot F_{n+1}.$$

Proof: When $n = 0$ it is true because $F_0^2 = F_0 \cdot F_1 = 0$. Assume

$$\sum_{i=0}^{n} F_i^2 = F_n \cdot F_{n+1}.$$

Then

$$\begin{aligned}
\sum_{i=0}^{n+1} F_i^2 &= \left(\sum_{i=1}^{n} F_i^2\right) + F_{n+1}^2 \\
&= (F_n \cdot F_{n+1}) + F_{n+1}^2 && \text{(by induction)} \\
&= F_{n+1}\left(F_n + F_{n+1}\right) \\
&= F_{n+1} \cdot F_{n+2} && \text{(by definition).}
\end{aligned}$$

d)

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

Proof: When $n = 1$ it is true because $F_0F_2 - F_1^2 = (-1)^1 = -1$. Assume

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

Note that, because $F_{n-1} + F_n = F_{n+1}$, $F_n - F_{n+1} = -F_{n-1}$. Then

$$\begin{aligned}
F_nF_{n+2} - F_{n+1}^2 &= F_n\left(F_n + F_{n+1}\right) - F_{n+1}^2 && \text{(by definition)}\\
&= F_n^2 + F_{n+1}\left(F_n - F_{n+1}\right)\\
&= F_n^2 - F_{n+1}F_{n-1} && \text{(by our note)}\\
&= -\left(F_{n-1}F_{n+1} - F_n^2\right) = -\left((-1)^n\right) && \text{(by induction)}\\
&= (-1)^{n+1}.
\end{aligned}$$

3. GRADING EXERCISE: Quiz 2

VERSION 1: Without using that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0,$$

prove that the number of subsets of $[n]$ of even size and the number of subsets of $[n]$ of odd size are equal.

Proof: We establish an invertible function between the set of even subsets of $[n]$ and the set of odd subsets of $[n]$ (recall that this just means a rule which establishes a one-to-one correspondence between the two sets, and a one-to-one correspondence between two finite sets means they have the same size). Consider any subset of $[n]$, and consider the element 1. It is either an element of the set or it is not. The function is made according to the following rule: if 1 is in the subset, take it out, and if it isn't, add it in. This changes the cardinality of a subset by exactly one, so it maps even subsets to odd and vice versa. Finally, this rule allows us to determine what to do with every subset, and there is no ambiguity.
Alternate Proof: By induction (given in class).

VERSION 2: Prove the following inequalities for $n \geq k \geq 0$ (with $n, k \in \mathbb{Z}$, and using that $0^0 = 1$):

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!}.$$

Proof: We prove the right-hand inequality first: it is easier.

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-(k-1))}{k!} \\
&\leq \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{k!} \\
&= \frac{n^k}{k!}.
\end{aligned}$$

For the second, we will use the fact that, if $0 \leq x < k \leq n$, then $\frac{n}{k} \leq \frac{n-x}{k-x}$. Then:

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-(k-1))}{k \cdot (k-1) \cdot (k-2) \cdot \cdots \cdot (k-(k-1))} \\
&= \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \frac{n-2}{k-2} \cdot \cdots \cdot \frac{n-(k-1)}{k-(k-1)} \\
&\geq \frac{n}{k} \cdot \frac{n}{k} \cdot \frac{n}{k} \cdot \cdots \cdot \frac{n}{k} \\
&= \left(\frac{n}{k}\right)^k.
\end{aligned}$$

6. a) The ways of counting anagrams can be summed up as follows: For a collection of n total letters, of k distinct kinds, where there are a_i of each letter of kind i (so $0 \le a_i \le n$ and $\sum a_i = n$), the number of anagrams is

$$\frac{n!}{a_1!a_2!\dots a_k!}.$$

Prove that if $k = 2$, this is just a binomial coefficient (it is $\binom{n}{a_1} = \binom{n}{a_2}$).

Proof: If $k = 2$, we have $a_1 + a_2 = n$, or $a_2 = n - a_1$, and so

$$\frac{n!}{a_1!a_2!} = \frac{n!}{a_1!(n-a_1)!} = \binom{n}{a_1}.$$

b) Because of the relation to binomial coefficients, these are called *multinomial coefficients* and denoted

$$\binom{n}{a_1\ a_2\ \dots\ a_k}.$$

Or is it because of that? Generalize the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

to discover the an expression of the form

$$(x_1 + x_2 + \dots + x_k)^n = ???$$

Solution: When multiplying out $(x_1+x_2+\dots+x_k)^n$, each term is made by choosing x_1 from some factors, x_2 from some factors, and so on. Each term will have exactly n x_i's, and so it can be written as

$$x_1^{a_1} x_2^{a_2} \dots x_k^{a_k},$$

with

$$0 \le a_i \le n \text{ and } \sum a_i = n.$$

The number of these terms in the expansion is then the multinomial coefficient $\binom{n}{a_1\ a_2\ \dots\ a_k}$, and so it's tricky to write, but we have the so-called *multinomial theorem*:

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{\substack{0 \le a_i \le n \\ a_1+a_2+\dots+a_k=n}} \binom{n}{a_1\ a_2\ \dots\ a_k} x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}.$$

7. We have a pile of $n > 1$ stones. Split the pile into two non-empty piles, and call the product of the two sizes p_1. Now pick one of the two piles and do the same (split it, then multiply the two sizes together to get a number called p_2). Keep repeating, $n-1$ times, until you have n piles with one stone each (note, then, that p_{n-1} will always be 1). Use induction to prove that, no matter how this sequence of splits is done,

$$\sum_{i=1}^{n-1} p_i = \binom{n}{2}.$$

Proof: When $n = 2$ this is true because $p_1 = 1 \cdot 1 = 1 = \binom{2}{2}$. Assume for strong induction that

$$\sum_{i=1}^{k-1} p_i = \binom{k}{2}$$

for all piles of $0 < k < n$ stones. So for n stones, our first split is into two piles, of sizes k and $n-k$ (both $< n$), and so

$$p_1 = k(n-k).$$

To finish this process, we then need to split our pile of size k all the way down to k piles of size 1. By induction the contribution of these splits to $\sum_{i=2}^{n-1} p_i$ is $\binom{k}{2}$. Similarly, the contribution of the $n-k$ pile is $\binom{n-k}{2}$. But then

$$\sum_{i=1}^{n-1} p_i = k(n-k) + \binom{k}{2} + \binom{n-k}{2} = \binom{n}{2}.$$

REMINDER: These represent possible solutions to each problem. The solution methods are not necessarily unique, and there are likely other correct solutions.