# Exam 2: Math 314

**Don't forget your name:** ____________________________ **Score:** _______ / **100**

**Instructions: No calculators.** Show all work to receive credit. Partial credit is available, but random scribbles or unexplained answers are worth zero. Leave your answer exact and **unsimplified** (i.e. $\binom{10}{4}$, not 210), and give numerical answers where applicable. Remember to reread the question over in between each part, so that you remember where you are going.

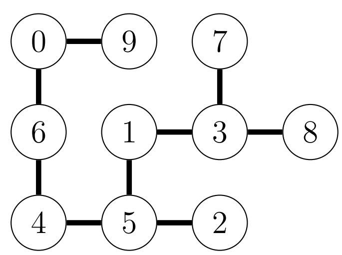
## Part 1: Short Answer. Show all your work, and carefully justify your answers.

[10] 1. Draw the labeled tree on 10 vertices with Prüfer code 5 3 3 1 5 4 6 0.

**Solution:** The second row of the extended Prüfer code is 533154600. We can then use the typical algorithm to recover the first row, and we get

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 7 | 8 | 3 | 1 | 5 | 4 | 6 | 9 |
| 5 | 3 | 3 | 1 | 5 | 4 | 6 | 0 | 0 |

as our list of edges. We can then read this list backwards to quickly construct our tree:




[10] 2. The degree sequence of a certain tree is $6, 5, 4, 3, 2, 1, \ldots, 1$. How many 1's are in the sequence? (HINT: How many edges does the tree have? How many vertices?

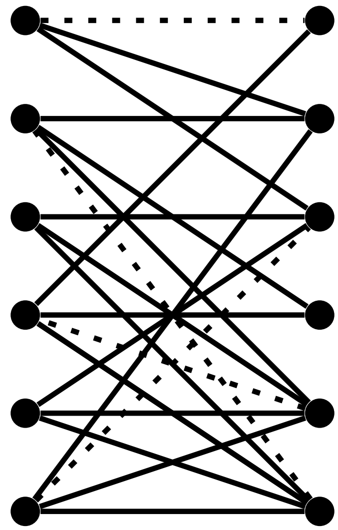
**Solution:** If there are $k$ 1's in the sequence, then there are $k+5$ vertices in the tree, and so there are $k+4$ edges. The sum of the degrees is $6+5+4+3+2+k = 20+k$. Then, because the sum of the degrees is twice the number of edges, we know

$$20 + k = 2(k + 4),$$

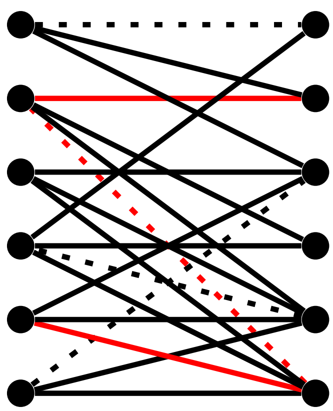
or

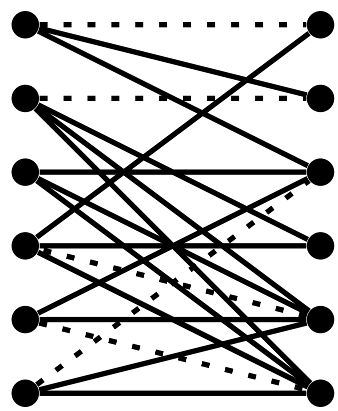
$$k = 12.$$

[10] 3. A partial matching of the following graph is given by the dashed edges. Use the augmenting path algorithm to extend this matching to a perfect matching. (Giving a matching without describing the algorithmic steps used to get it is worth 0 points)

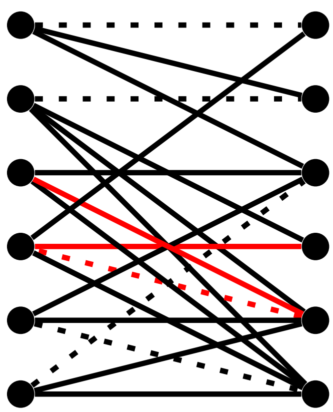


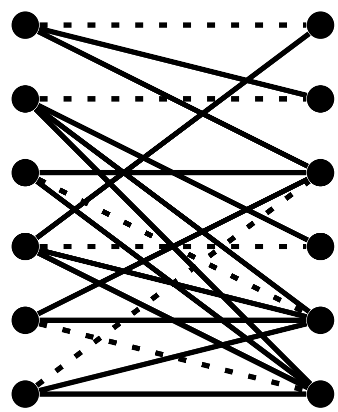
**Solution:** There are plenty of ways to do this. Here's one. An augmenting path, and the partial matching after augmentation:





One more, and the perfect matching it gives:





[30] 4. For each of the following classes of graphs, give a property of graphs satisfied by *every* graph in that class. However, you may not list the same property for more than one part. Remember what a property is and isn't.

(a) Trees

> **Solution:** Trees are connected, acyclic, 2-colorable, and planar, but that's about it.

(b) Complete graphs

> **Solution:** Complete graphs are connected, Hamiltonian, diameter 1, radius 1, chromatic number $n$, a few other things...

(c) Bipartite graphs

> **Solution:** About the only things you can say about them *all* is that they're 2-colorable (not chromatic number 2, if there aren't edges it's 1).

(d) Cycles

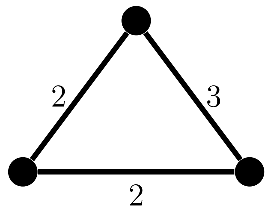
> **Solution:** They're all connected, 2-connected, planar, 3-colorable, Eulerian, Hamiltonian, some other stuff...

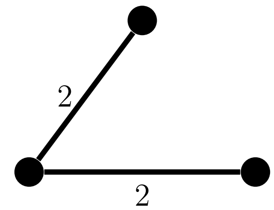
(e) Stars

> **Solution:** They're a special class of trees, so any of those answers.

[10] 5. Draw an edge-weighted graph with positive integer weights such that, when you use the greedy algorithm to find the minimum weight spanning tree, there are two vertices where the cheapest path between them is not in the tree. (HINT: you really don't need that many vertices)

**Solution:** Below is a graph and its minimum weight spanning tree:



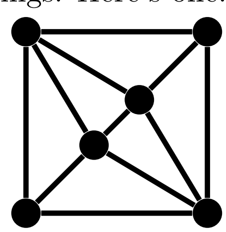




and the path in the tree from the top to the right has weight 4, while there is a path of weight 3 in the graph.

[10] 6.

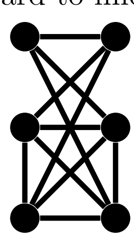
[5] (a) Draw a planar graph with degree sequence $4, 4, 4, 4, 3, 3$.

**Solution:** There are plenty of drawings. Here's one:



[5] (b) Draw a nonplanar graph with degree sequence $4, 4, 4, 4, 3, 3$.

**Solution:** We know that to be be non-planar, $K_{3,3}$ will have to be a subgraph (there is a solution where $K_5$ is a minor, but it's hard to find). Then we don't have much choice:



## Part 2: True/False.

[20] 7. Conceptual Questions: True or False? Write the WHOLE WORD below the question. You do not need to justify your answer.

[5] (a) A (simple) graph that is regular of degree $d > 0$ has chromatic number at least $d$.

**Solution: False**. $K_{d,d}$ has chromatic number 2, no matter what $d$ is.

[5] (b) Deleting a vertex from a graph cannot increase connectivity.

**Solution: False**. A cycle with a leaf hanging off it has a cut edge, but if you delete the leaf, the cycle is 2-connected.

[5] (c) All regular graphs (with at least one edge) are connected.

**Solution: False**. Two triangles, for example.

[5] (d) Every graph with fewer edges than vertices is planar.

**Solution: False**. The graph formed with $K_5$ and, say, $10,000,000$ isolated vertices. If I had included the word connected, it would be true.