Non-Transitive Dice and Directed Graphs

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Outline

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- The concept is (perhaps) best explained in terms of a game.
- I will take this concept and extrapolate it to a different setting (directed graphs).

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- In the above example, $P(A \succ B) = \frac{21}{36}$, $P(B \succ C) = \frac{21}{36}$, and $P(C \succ A) = \frac{25}{36}$.

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- This condition will be called *balance*.
- $\begin{array}{lcccccc} A: & 18 & 14 & 11 & 7 & 4 & 3 \\ B: & 17 & 13 & 10 & 9 & 6 & 2 \\ C: & 16 & 15 & 12 & 8 & 5 & 1 \end{array}$ has victorious probability $\frac{19}{36}$.

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- A BntD(4): $\begin{array}{ccccc} A: & 12 & 10 & 3 & 1 \\ B: & 9 & 8 & 7 & 2 \\ C: & 11 & 6 & 5 & 4 \end{array}$, v.p. $\frac{9}{16}$.

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- A BntD(5): $\begin{matrix} A: & 15 & 11 & 7 & 4 & 3 \\ B: & 14 & 10 & 9 & 5 & 2 \\ C: & 13 & 12 & 8 & 6 & 1 \end{matrix}$, v.p. $\frac{13}{25}$.

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Concatenation Lemmas

- Concatenation of two sets of dice:

$$D_\sigma = \begin{cases} A: & 9 \quad 5 \quad 1 \\ B: & 8 \quad 4 \quad 3 \\ C: & 7 \quad 6 \quad 2 \end{cases}, \quad D_\rho = \begin{cases} A: & 9 \quad 4 \quad 2 \\ B: & 8 \quad 6 \quad 1 \\ C: & 7 \quad 5 \quad 3 \end{cases}.$$

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- $$D_{\rho\sigma} = \left\{ \begin{array}{lccc|ccc} A: & 18 & 14 & 10 & 9 & 4 & 2 \\ B: & 17 & 13 & 12 & 8 & 6 & 1 \\ C: & 16 & 15 & 11 & 7 & 5 & 3 \end{array} \right. .$$

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Lemmas (S. 2011)

Concatenation preserves both balance and non-transitivity.

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Existence Theorem

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There exists a BntD(n) for every $n \geq 3$.

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- By lemmas, concatenation of two BntD's is a BntD.

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A General Result

Theorem (S.2011)

Let A, B, C be disjoint subsets of $[3n]$, each of size n, and let P_{ab} represent the probability that a randomly chosen element of A is larger than a randomly chosen element of B (also analogously define P_{bc} and P_{ca}). Then

$$P_{ab} = P_{bc} = P_{ca}$$

if and only if

$$\sum_{a \in A} a = \sum_{b \in B} b = \sum_{c \in C} c.$$

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- The statement then says we can build a set of dice that *realizes* the edges of this directed graph.
- Next natural question: more vertices (more dice)?

More Dice?

- Efron's Dice: $\begin{array}{lcccccc} A: & 4 & 4 & 4 & 4 & 0 & 0 \\ B: & 3 & 3 & 3 & 3 & 3 & 3 \\ C: & 6 & 6 & 2 & 2 & 2 & 2 \\ D: & 5 & 5 & 5 & 1 & 1 & 1 \end{array}.$

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- Expand to:

$$\begin{array}{lcccccc} A: & 19 & 18 & 17 & 16 & 2 & 1 \\ B: & 15 & 14 & 13 & 12 & 11 & 10 \\ C: & 24 & 23 & 9 & 8 & 7 & 6 \\ D: & 22 & 21 & 20 & 5 & 4 & 3 \end{array}$$

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- Suitably modify definitions to: $\mathrm{BntD}_m(n)$.

More more dice?

- $\text{BntD}_4(3)$: $\begin{array}{llll} A: & 12 & 5 & 2 \\ B: & 11 & 8 & 1 \\ C: & 10 & 7 & 3 \\ D: & 9 & 6 & 4 \end{array}$,

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- $\text{BntD}_4(5)$: $\begin{array}{lccccc} A: & 20 & 13 & 10 & 6 & 4 \\ B: & 19 & 15 & 9 & 8 & 3 \\ C: & 18 & 16 & 12 & 5 & 1 \\ D: & 17 & 14 & 11 & 7 & 2 \end{array}$.

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- Concatenation arguments. . .

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- Proof: By induction.

Induction Example

- $$\begin{array}{llll} A: & 9 & 5 & 1 \\ B: & 8 & 4 & 3 \\ C: & 7 & 6 & 2 \end{array} \mapsto \begin{array}{llll} A: & \hat{9} & \hat{5} & \hat{1} \\ B: & \hat{8} & \hat{4} & \hat{3} \\ C: & \hat{7} & \hat{6} & \hat{2} \\ D: & ? & ? & ? \end{array}$$

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- So, because $C \succ A$, we step outside $\mathbb{N}$ to make D numerically "similar" to C, but "inferior" to it:

  $$\begin{array}{lccc} A: & 9 & 5 & 1 \\ C: & 7 & 6 & 2 \\ D: & 6.9 & 5.9 & 2.1 \end{array}$$

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- Finally, re-label linearly:

A :	9	5	1		A :	12	6	1
B :	8	4	3		B :	11	5	4
C :	7	6	2	$\mapsto$	C :	10	8	2
D :	6.9	5.9	2.1		D :	9	7	3

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- We then shift the labels of die $K + 1$. First, a few notes.

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- Then there do in fact exist labels of $K+1$ providing the necessary relationship (from a purely $\succ$ viewpoint).
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- We will now, however, recover it.

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- And so on...
- *Any number of victories we would desire is obtainable!*

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- *Tournament*: a copy of the complete graph on m vertices K_m, where each edge is given an orientation from one incident vertex to the other.
- Given an arbitrary tournament, does it have a *realization* as dice? Balanced dice??
- Because acyclic graphs correspond to total (well) orderings, they are trivially realizable, even by one-sided dice. However, the condition of balance (all victorious probabilities equal) has no meaning here.

A Few Cases Answered

- Any tournament T_3 is either acyclic, which is realizable, or it is a directed 3-cycle (a $BntD(n)$).

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  - One is a 3-cycle, plus a “loser”✓
  - Only one has a 4-cycle, so that is a $BnD_4(n)$✓

A Few Cases Answered

- The real issue…

Outline

The Ignored Structure

- With 4 dice,
-

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- $A \succ B \succ C \succ D \succ A$.
- How do A, C relate? B, D?

The Ignored Structure

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- $$\begin{array}{rccc} A: & 12 & 6 & 1 \\ B: & 11 & 5 & 4 \\ C: & 10 & 8 & 2 \\ D: & 9 & 7 & 3 \end{array}$$
- $A \succ B \succ C \succ D \succ A$.
- How do A, C relate? B, D?
- Is it forced or can it be manipulated?

The New Question

- Can an arbitrary orientation of K_n be constructed with a set of dice?

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- Can an arbitrary orientation of K_n be constructed with a set of dice?
- If so, we will call the given tournament *realizable*.

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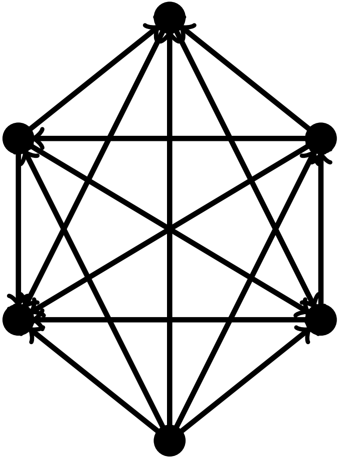
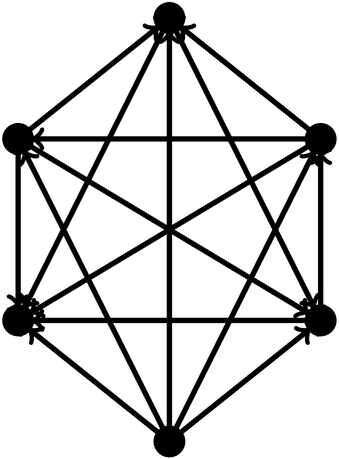
- Alternately: a connected component is a set of vertices which form a maximal directed cycle.
- Belonging to a connected component is an equivalence relation.
- Viewing a component as one new vertex, and keeping all edges not contained in a component gives an acyclic directed graph.

Example



Example



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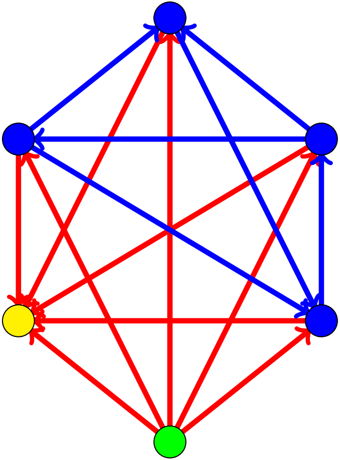
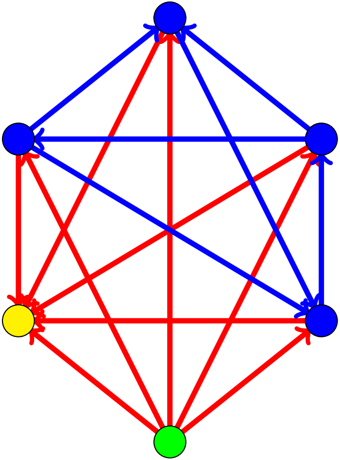
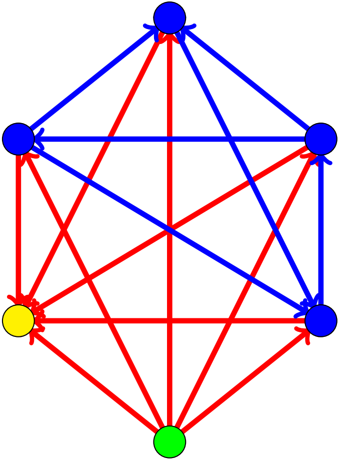


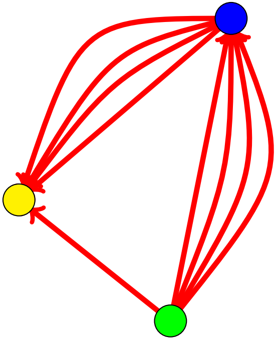
Example



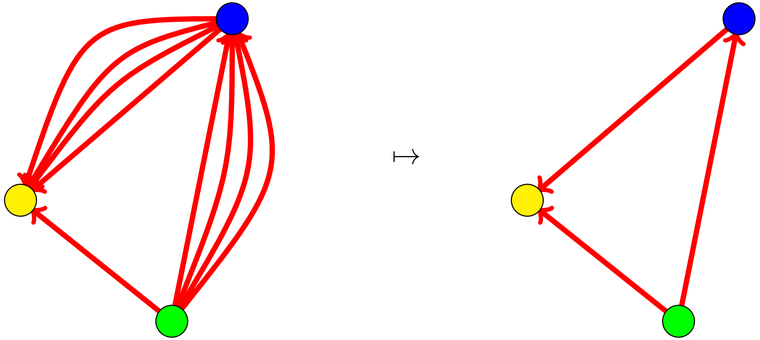
Example



$\mapsto$



Example



Strong Tournaments

- A tournament is said to be *strong* if it has exactly 1 connected component.

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- Alternately: a tournament is strong if and only if it contains a directed Hamilton cycle.
- Note: a strong tournament then has a subgraph (obtained only by deleting edges) which is a directed cycle and thus can be realized with a set of balanced dice.
- We now recover the deleted edges.

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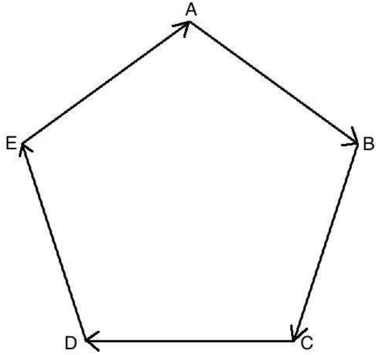
- Then, we will check the number of victories for the pair of dice we wish to add as an edge in our tournament.
- Then, we will add sides to those dice to alter the number of victories (or to ensure all dice end up with the same number of sides).

Algorithm Example

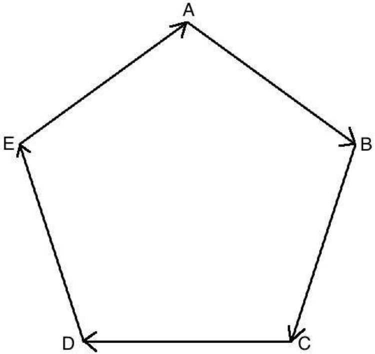
We start with a set of 5 dice. Note that every die has either 4 or 5 victories over any other.



	A :	15	7	1
	B :	14	6	5
$\rightarrow$	C :	13	10	2
	D :	12	9	3
	E :	11	8	4

Algorithm Example

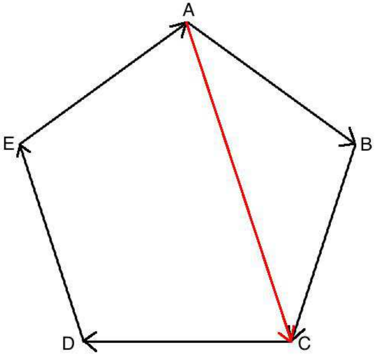
Using a little foresight, we shift our labels up by 10.



	A :	25	17	11
	B :	24	16	15
$\rightarrow$	C :	23	20	12
	D :	22	19	13
	E :	21	18	14

Algorithm Example

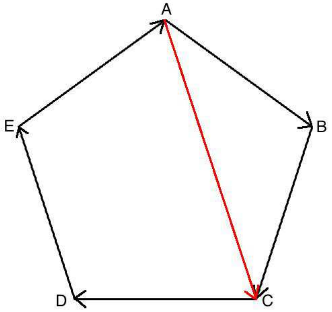
The first edge we choose to add requires $A \succ C$.



	A :	?	25	17	11	?
	B :		24	16	15	
$\rightarrow$	C :	?	23	20	12	?
	D :		22	19	13	
	E :		21	18	14	

Algorithm Example

$A \succ C$ means we place the larger of $\{26, 27\}$ on A. A only had 4 victories over C, so we place the larger of $\{9, 10\}$ on A to add one more.

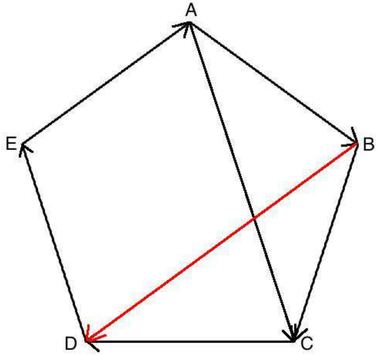


	A :	27	25	17	11	10
	B :		24	16	15	
$\rightarrow$	C :	26	23	20	12	9
	D :		22	19	13	
	E :		21	18	14	

You might worry how this affects the relationship between say, A and B. We'll discuss that in a moment.

Algorithm Example

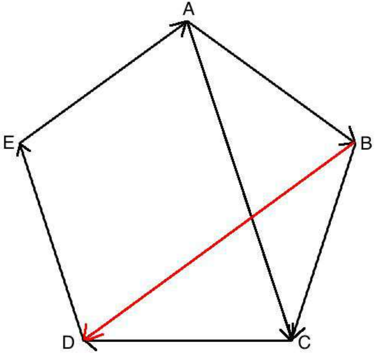
The next edge requires that $B \succ D$.



A :	27	25	17	11	10
B :	?	24	16	15	?
$\rightarrow$ C :	26	23	20	12	9
D :	?	22	19	13	?
E :		21	18	14	

Algorithm Example

$B \succ D$ means we place the larger of $\{28, 29\}$ on B. This time, B already had 5 victories over D, so we place the *smaller* of $\{7, 8\}$ on B.

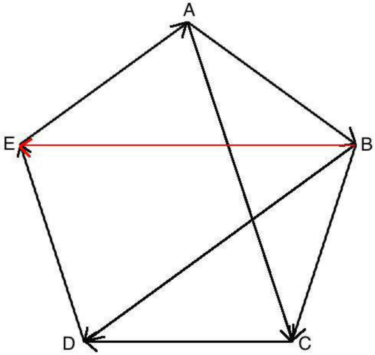


	A :	27	25	17	11	10
	B :	29	24	16	15	7
$\rightarrow$	C :	26	23	20	12	9
	D :	28	22	19	13	8
	E :		21	18	14	

Now, look back to A and B.

Algorithm Example

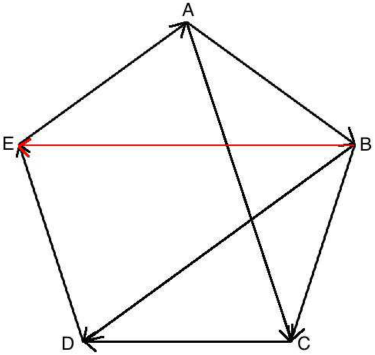
The next edge we choose is $B \succ E$.



$$\rightarrow \begin{array}{llc|ccc|cc} A: & & 27 & 25 & 17 & 11 & 10 & \\ B: & ? & 29 & 24 & 16 & 15 & 7 & ? \\ C: & & 26 & 23 & 20 & 12 & 9 & \\ D: & & 28 & 22 & 19 & 13 & 8 & \\ E: & & ? & 21 & 18 & 14 & ? & \end{array}$$

Algorithm Example

$B \succ E$ means we place the larger of $\{30, 31\}$ on B. B originally had 5 victories over E, so we place the smaller of $\{5, 6\}$ on B.

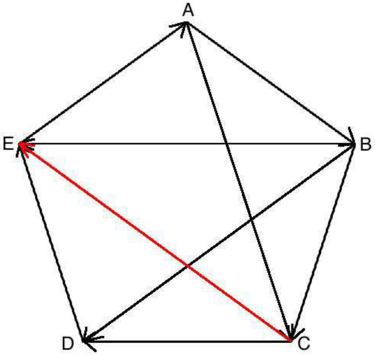


$\rightarrow$

A :		27	25	17	11	10	
B :	31	29	24	16	15	7	5
C :		26	23	20	12	9	
D :		28	22	19	13	8	
E :		30	21	18	14	6	

Algorithm Example

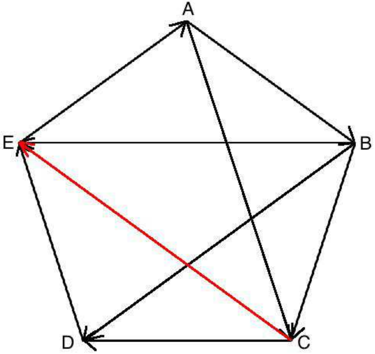
The next edge we add is $C \succ E$.



	A :		27	25	17	11	10	
	B :	31	29	24	16	15	7	5
$\rightarrow$	C :	?	26	23	20	12	9	?
	D :		28	22	19	13	8	
	E :	?	30	21	18	14	6	?

Algorithm Example

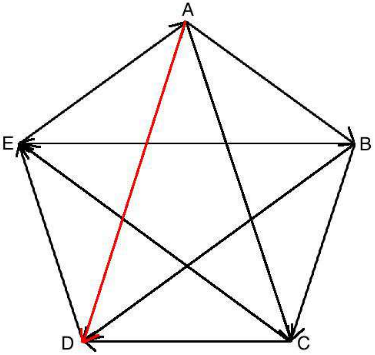
$C \succ E$ means we place the larger of $\{32, 33\}$ to C. C had 5 victories over E, so give it the smaller of $\{3, 4\}$.



A :		27	25	17	11	10	
B :	31	29	24	16	15	7	5
$\rightarrow$ C :	33	26	23	20	12	9	3
D :		28	22	19	13	8	
E :	32	30	21	18	14	6	4

Algorithm Example

Lastly, we add $A \succ D$.



$\rightarrow$

A :	?	27	25	17	11	10	?
B :	31	29	24	16	15	7	5
C :	33	26	23	20	12	9	3
D :	?	28	22	19	13	8	?
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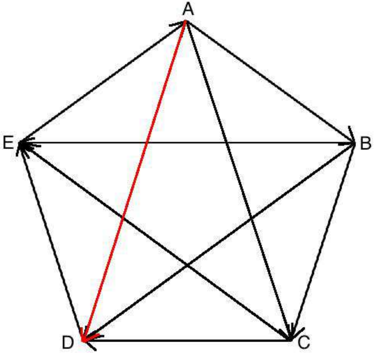
Algorithm Example

$A \succ D$ means A gets the larger of $\{34, 35\}$. A had 4 victories over D, so it gets the larger of $\{1, 2\}$.

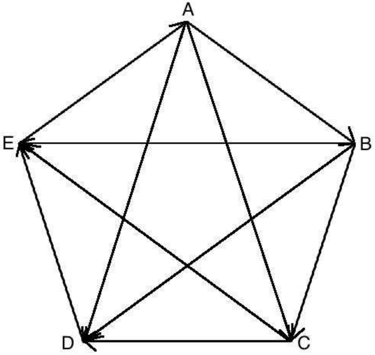


$\rightarrow$

A :	35	27	25	17	11	10	2
B :	31	29	24	16	15	7	5
C :	33	26	23	20	12	9	3
D :	34	28	22	19	13	8	1
E :	32	30	21	18	14	6	4

Algorithm Example

We have constructed a set of balanced dice such that the probabilities correspond exactly with the direction of the edges.



$\rightarrow$

A:	35	27	25	17	11	10	2
B:	31	29	24	16	15	7	5
C:	33	26	23	20	12	9	3
D:	34	28	22	19	13	8	1
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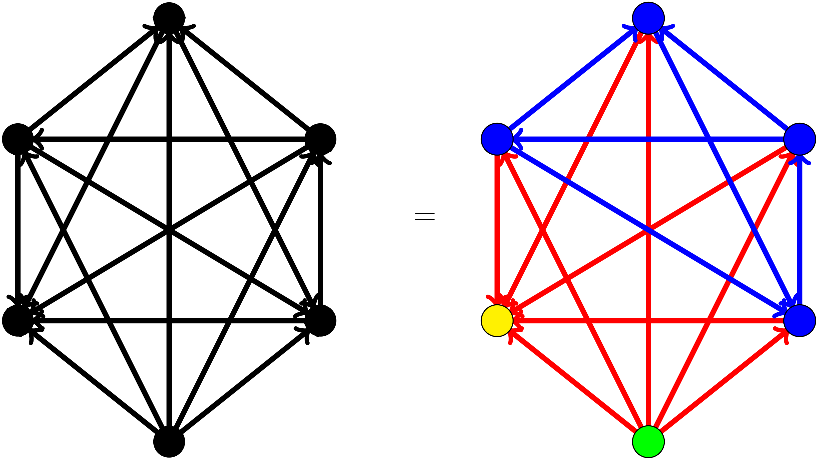
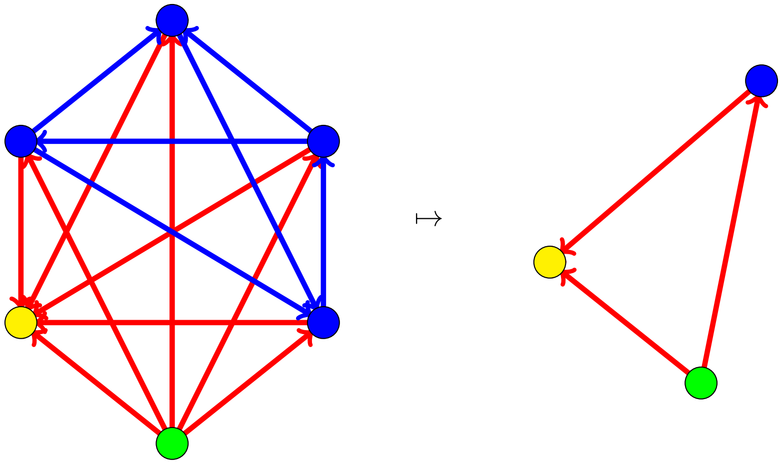
Non-Strong Tournaments

- If a tournament is not strong, *it is acyclic on its strong components*.
- Then, do the following:
  1. Take the subgraphs consisting of each strong component.
  2. Perform the above algorithm on them.
  3. Concatenate in acyclic (total) order.

Non-Strong Tournaments



Non-Strong Tournaments



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Balance In Tournaments

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- So, if the tournament is strong, it can be realized with *balanced* dice.
- If the tournament is not strong, balance is impossible.

Incomplete Directed Graphs

- An arbitrary directed graph isn't a problem.

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- An arbitrary directed graph isn't a problem.
- Consider it as a subgraph of a tournament on the same vertex set.

The End Result

Theorem (S. 2012):

Let G be a directed graph. Then there is a set of dice which realizes G. Further, the dice can be made balanced if and only if G is a subgraph of a strong tournament on the same vertex set.

Thank You!