Classifying 2-Transitive Perfect Matchings

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Outline

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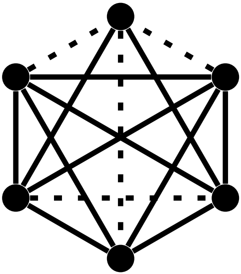
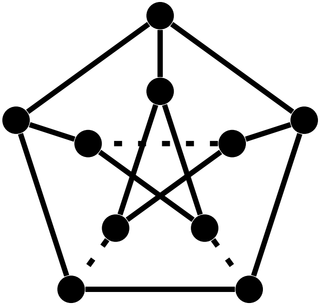
- $\sigma : E \mapsto \{+, -\}$ is a function.

Some Signed Graphs





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- A graph in which every cycle is balanced is a balanced signed graph.

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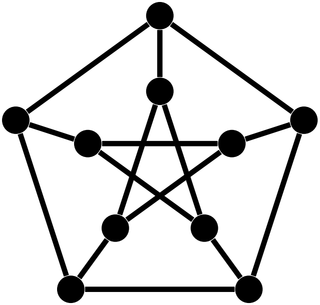

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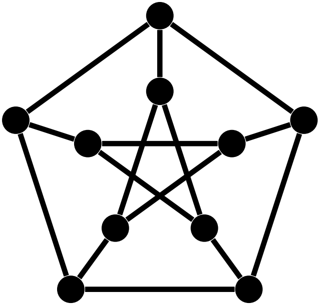
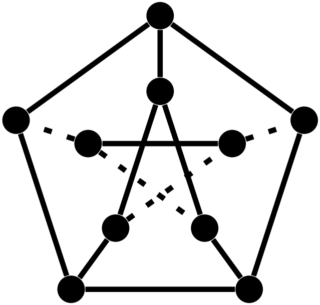
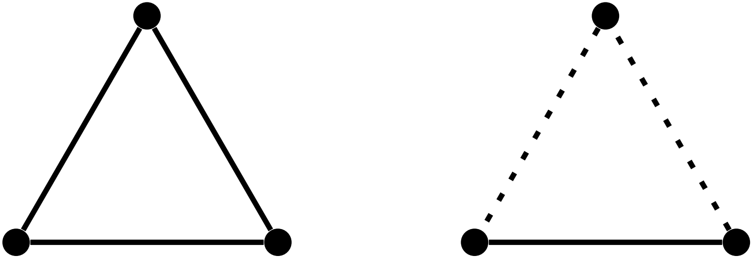




Balanced Triangles



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- We can then partition the set of signings of an underlying graph into *switching* (isomorphism) *classes*.

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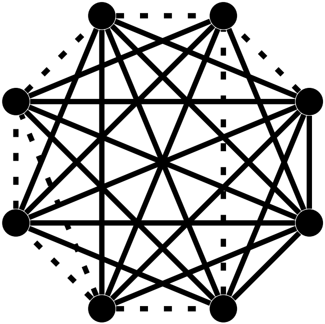
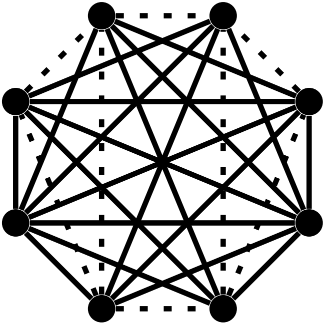
- Question: does this vector uniquely identify a switching class of K_n?

Sorry, But No





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A Conjecture

- Make a large matrix where every row is the negative cycle vector of some switching class.

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Theorem (S. 2015)

This matrix has full column rank for K_n and $K_{m,n}$.

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- The collection that appears to work for almost everything:
- A maximum *negative matching* and it's submatchings
- However...

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Definition

- Let $\mathcal{M}$ be a matching in a graph Γ with m edges, and $G \leq \mathrm{Aut}(\Gamma)$. Say $\mathcal{M}$ is *G-permutable* if $G^{E(\mathcal{M})} \cong S_m$.

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- If such an $\mathcal{M}$ exists, say Γ contains an m-permutable matching.
- The computations done previously only work on permutable matchings!

Some Results

Proposition (S., Swartz (2016))

Let Γ be a *G*-arc-transitive graph with a *G*-permutable *m*-matching, where $m \geq 6$. If $\{\alpha, \beta\}$ is an edge of Γ, then there is a subgroup $U \leq G_{\alpha\beta}$ such that $U^{\Gamma(\alpha)}$ has a composition factor isomorphic to A_{m-1}.

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Corollary (S., Swartz (2016))

If Γ is a *G*-arc-transitive graph with a *G*-permutable *m*-matching, where $m \geq 6$, then the degree of the graph Γ is at least *m*.

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Theorem (S., Swartz (2016)

Let Γ be a connected graph with a perfect matching $\mathcal{M}$ (with m edges) such that Aut(Γ) is 2-transitive on $\mathcal{M}$. Then Γ is one of the following:

Case 1

A join between two graphs that are either complete or edgeless:

a) $K_m \vee K_m \cong K_{2m}$,
b) $K_m \vee \overline{K}_m$,
c) $\overline{K}_m \vee \overline{K}_m \cong K_{m,m}$

Case 2

A *matching join* between two graphs that are either complete or edgeless (but not both edgeless):

a) $K_m \underline{\vee} K_m$,

b) $K_m \underline{\vee} \overline{K}_m$,

Case 3

Let $m = p^f \equiv 3 \mod 4$ with p prime

a) Γ is the incidence graph of the Paley symmetric 2-design over $GF(p^f)$,

b) Γ is the graph obtained from the incidence graph of the Paley symmetric 2-design over $GF(p^f)$ by replacing the independent sets with copies of K_{p^f}

(Note for *a*: the points of this design are elements of $GF(p^f)$ and the blocks are the translates of the set of nonzero squares, i.e. $V(\Gamma) = GF(p^f) \times \{0, 1\}$ and $(x, i), (y, j) \in V(\Gamma)$ are adjacent iff $i = 0$, $j = 1$, and $y - x$ is a square in $GF(p^f)$)

Case 4

Let $m = 5$

a) Γ is the Petersen graph,

b) $\Gamma = K_{10} \setminus \{2 \cdot C_5\}$, where $V(\Gamma) = [10]$ and the two removed cycles are $(1, 2, 3, 4, 5)$ and $(6, 8, 10, 7, 9)$.

Thank You!