Extra Credit: More on Sequences

Due at the beginning of class on or before November 11th

One of the most interesting types of sequences is a bit more complicated, and as such we did not cover it deeply in class. It is called a *recursive* sequence.

To define something recursively essentially means to define it in terms of "what we've already done". So, in terms of sequences, we define things according to a rule. We've already seen this: for example, the sequence

$$1, 2, 3, 4, 5, \ldots$$

is easily seen to be the sequence

$$a_n = n.$$

The difference with a recursive sequence is that the rule is not just given in terms of n (such a rule is called *explicit*). Instead, it may be given in terms of the previous terms in the sequence. Here is an example:

$$a_{n+1} = 1 + a_n.$$

This says that any term in the sequence is simply 1 more than the term before it. So, given a term, we can find the next term (just add one). But there is a problem with this: how does it begin? In order to fix this, we need to include some more information. This will usually be the first term (or terms) of the sequence:

$$a_1 = 1,\ a_{n+1} = 1 + a_n.$$

Let's write out a few terms of this sequence:

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= 1 + a_1 = 1 + 1 = 2 \\
a_3 &= 1 + a_2 = 3 \\
a_4 &= 1 + a_3 = 4 \\
&\vdots
\end{aligned}$$

And so on. The terms are $1, 2, 3, 4, 5, \ldots$ *just as before!* What this demonstrates is that there is often a recursive formula for a sequence with an explicit formula.

Exercise 1: For each explicit formula below, write a recursive formula.

(a) $a_n = 2n + 1$ (the odd numbers).

(b) $a_n = n!$ (the factorial function).

(c) $a_n = 2^n$.

(d) $a_n = n^3$.

(e) $a_n = \frac{1}{n}$ (hint: write out a few terms, try to find the relationship between them).

Here is a more complicated recursive sequence:

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1}.$$

Let's write out a few terms of this sequence:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_3 &= a_1 + a_2 = 1 + 1 = 2 \\ a_4 &= a_2 + a_3 = 3 \\ a_5 &= 5 \\ a_6 &= 8 \\ &\vdots \end{aligned}$$

Finding an explicit formula for this sequence is difficult. Ask if you're interested, but it would take too long here. I will mention that it is

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Yuck. The advantage to the explicit formula, as ugly as it is, is plain: what is a_{100}? The recursive formula requires 100 calculations. With the explicit formula, plug in $n = 100$ and you're done. The point then, is that, nasty or not, an explicit formula also has its uses.

Exercise 2: For each recursive formula below, write an explicit formula.

(a) $a_1 = 2,\ a_{n+1} = 2 + a_n$.

(b) $a_1 = 1,\ a_{n+1} = n \cdot a_n$.

(c) $a_1 = 1,\ a_{n+1} = \left(\frac{1}{n+1}\right) \cdot a_n$.

(d) $a_1 = 1,\ a_{n+1} = 2 \cdot a_n$.

(e) $a_1 = 4,\ a_{n+1} = \sqrt{a_n}$.

Now let's discuss convergence. With the example from above, we have two ways of writing this sequence:

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1},\ \text{and } a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

The second way, the explicit formula, as we have already mentioned, is good for a few things. But, what if we are faced with the question "does this sequence converge or diverge?" We would have to take the limit of our formula. No thanks. However, with the recursive formula, it is pretty easy to see that the terms are getting bigger, and do so without bound, so we can say the sequence diverges.

Exercise 3: For each recursive formula below, decide whether the sequence converges or diverges.

(a) $a_1 = 2,\ a_{n+1} = 2 + a_n$.

(b) $a_1 = 1,\ a_{n+1} = 2 \cdot a_n$.

(c) $a_1 = 1,\ a_{n+1} = \left(\frac{1}{n+1}\right) \cdot a_n$.

(d) $a_1 = 1,\ a_{n+1} = \sqrt{a_n}$.

(e) $a_1 = 4,\ a_{n+1} = \sqrt{a_n}$.

(f) $a_1 = 1,\ a_2 = 2,\ a_{n+2} = \frac{a_n}{a_{n+1}}$.

Finally, let's try to find some limits of convergent recursive sequences. I'll do an example:

$$a_1 = 1,\ a_{n+1} = \sqrt{6 + a_n}.$$

First: all the terms are positive. We can see this as a_{n+1} is defined by a square root, so it returns a positive number. Then, I notice that there seems to something special about the number 3. That is, if

$$a_n = 3 \text{ for some } n,\ \text{ then } a_{n+1} = 3,$$

and all the terms after that are also equal to 3. Even better, if

$$a_n < 3,\ \text{ then } a_{n+1} < 3,$$

and if

$$a_n > 3,\ \text{ then } a_{n+1} > 3.$$

Exercise 4: Explain the above phenomenon.

Since $a_1 = 1 < 3$, it follows that every term of the sequence is less than 3. Now,

$$\begin{aligned} a_{n+1} - a_n &= \sqrt{6 + a_n} - \sqrt{6 + a_{n-1}} \\ &= \frac{\left(\sqrt{6 + a_n} - \sqrt{6 + a_{n-1}}\right) \cdot \left(\sqrt{6 + a_n} + \sqrt{6 + a_{n-1}}\right)}{\sqrt{6 + a_n} + \sqrt{6 + a_{n-1}}} \\ &= \frac{a_n - a_{n-1}}{\sqrt{6 + a_n} + \sqrt{6 + a_{n-1}}}. \end{aligned}$$

Because the denominator is positive, $a_{n+1} - a_n$ is positive if $a_n - a_{n-1}$ is positive. That is, the sequence is either always increasing, or it is always decreasing. Since

$$a_2 = \sqrt{7} > 1 = a_1,$$

the sequence is always increasing. Because it is increasing, but always less than 3, the sequence converges.

Exercise 5: Explain that statement.

So a_n has a limit. Calling it L, we have

$$\lim_{n\to\infty} a_n = L.$$

All that's left is to figure out what it is. So, because

$$a_{n+1} = \sqrt{6+a_n},$$

we have

$$\begin{aligned}
\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{6+a_n} \\
L &= \lim_{n\to\infty} \sqrt{6+a_n} \\
L &= \sqrt{\lim_{n\to\infty} (6+a_n)} \\
L &= \sqrt{6+L} \\
L^2 &= 6+L \\
L^2 - L - 6 &= 0 \\
(L-3)(L+2) &= 0
\end{aligned}$$

So our limit is either 3 or -2. But it cannot be negative (because a_n is always positive, so $L = 3$.

You might ask, "why can't we just do that last part to begin with?" The reason we can't is simple: unless a_n has a limit, we can't replace the limit with L. Replacing a limit with L is assuming the sequence converges. So, we actually have to show it converges. That's the point of Page 4.

Exercise 6: Show that, for the above example, a_n converges to 3 if $-6 < a_1 \leq 3$, and a_n diverges if $a_1 > 3$.

Exercise 7: For each recursive formula below, give the limit. Make sure to show the sequence converges.

(a) $a_1 = 1$, $a_{n+1} = \sqrt{42+a_n}$. (Hint: see above)

(b) $a_1 = 1$, $a_{n+1} = 1 - \frac{a_n}{3}$. (This converges, but showing it is tough)

(c) $a_1 = 1$, $a_{n+1} = \dfrac{1}{2^{a_n}}$. (I just made this up, but it converges. Good luck.)