Math 304 Section 8 Worksheet 2/26/16

Finding the inverse of a matrix

Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

To determine is A is invertible (and simultaneously find A^{-1} if A is invertible), we can try to solve the matrix equation $AX = I$. A is square, so if there is a matrix X that satisfies this equation, then A is invertible and $A^{-1} = X$. If there is no X that satisfies this equation, then A is not invertible. To solve this equation, you can find the reduced row echelon form of the augmented matrix

$$\left[\begin{array}{c:c} A & I \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Exercise 1. Find the reduced row echelon form of $\left[\begin{array}{c:c} A & I \end{array}\right]$.

When you're finished with the exercise, check to see that you have the matrix

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \end{array}\right]$$

Since none of the columns to right of the dashed line are pivot columns, there is a solution to $AX = I$. Therefore A is invertible. Moreover,

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

Exercise 2. Determine if $B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$ is invertible or not by finding a row echelon form of $\left[\begin{array}{c:c} B & I \end{array}\right]$.

If you put $\left[\begin{array}{c:c} B & I \end{array}\right]$ in row echelon form, you'll notice that there is a pivot column to the right of the dotted line. This means that the equation $BX = I$ has no solutions, and therefore B is not invertible.

Warning. Given any matrix M, it may be possible to find a solution to the equation $MX = I$ *even though M is not invertible!* The next exercise gives an example of this.

Exercise 3. Find all possible solutions to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You should have been able to find infinitely many solutions in the previous exercise. But if there is a matrix X such that $MX = I$, why isn't M invertible? (This would be a good time to open your textbook and remind yourself of the definition of the word *invertible*.)

Exercise 4. Open your textbook and read the definition of the word *invertible*.

Exercise 5. Explain why the matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ doesn't satisfy this definition.

You now know that finding a solution to $MX = I$ does *not* imply that M is invertible. So why are able to conclude that the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

is invertible just by finding a solution to $AX = I$?

Exercise 6. Think about why we are able to conclude that A is invertible.

Solution. (Don't read this until you've thought about the exercise.) The reason that we could conclude that A is invertible just by solving $AX = I$ is (seriously, stop reading here if you haven't done the exercise) because A is square. According to Theorem 2.7.8, a square matrix is invertible if and only if it is row equivalent to I. By row reducing $\left[\begin{array}{c:c} A & I \end{array}\right]$, you showed that A is row equivalent to I, and because A is square the theorem allows you to conclude that A is invertible.

In fact, according to Lemma 2.7.6, every invertible matrix is square. Therefore any matrix which is not square cannot be invertible. This gives another way to show that M is not invertible: just observe that it is not square.

Working with elementary matrices

Let's recall the elementary matrices. To make the examples easy to write, we'll only deal with 3×3 matrices.

Exercise 7. Switch any two rows of I_3. Write down the resulting matrix and call it E_1.

Exercise 8. Multiply any row of I_3 by a nonzero constant. Write down the resulting matrix and call it E_2.

Exercise 9. Add a multiple of one row of I_3 to a different row. Write down the resulting matrix and call it E_3.

You've written down three matrices. Each of these is an elementary matrix. (It doesn't matter what rows you chose to change). Moreover, every 3×3 elementary matrix can be obtained by performing exactly one elementary row operation on I_3.

The purpose of the next three exercises is to demonstrate what happens when you multiply an arbitrary matrix C on the left by an elementary matrix.

Let

$$C = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Exercise 10. Compute E_1C.

Exercise 11. Compute E_2C.

Exercise 12. Compute E_3C.

You should notice that E_1C is exactly the matrix you would get by switching two rows of C (the same rows you switched on I_3 in Exercise 7). Similarly, E_2C is the matrix you would get by multiplying a row of C by a nonzero constant (the same row and constant that you used in Exercise 8). Finally, E_3C is just the matrix you would get by adding a multiple of one row of C to another row of C (the same row operation you performed in Exercise 9).

It should be noted that every elementary matrix is invertible. There's also a very easy way to get their inverses.

Exercise 13. There is a row operation that undoes the row operation that you performed in Exercise 7. Perform this row operation on I_3. Write down the resulting matrix and call it F_1.

Exercise 14. Ditto for the row operation you performed in Exercise 8. Call the resulting matrix F_2.

Exercise 15. Ditto for the row operation you performed in Exercise 9. Call the resulting matrix F_3.

Exercise 16. Convince yourself that $F_1 = E_1^{-1}$, $F_2 = E_2^{-1}$, and $F_3 = E_3^{-1}$.

Exercise 17. Convince yourself that the inverse of any elementary matrix is an elementary matrix.

Now you should realize that *performing an elementary row operation on a matrix* is basically the same as *multiplying a matrix on the left by an elementary matrix*. Also, you can easily get the inverse of any elementary matrix E just by undoing the row operation that corresponds to E. This is a good segue into the next part of the worksheet.

Writing an invertible matrix as a product of elementary matrices

Recall again that an $n \times n$ matrix A is invertible if and only if it is row equivalent to I (this is part of Theorem 2.7.8). This means that there is a finite sequence of row operations that will turn A into I. According to what you now know about elementary matrices, this also means that there is a finite sequence of elementary matrices $E_1, E_2, \ldots, E_{k-1}, E_k$ such that

$$E_k E_{k-1} \ldots E_2 E_1 = I$$

You're going to work out an example of this in the next few exercises.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

(This is the invertible matrix from the beginning of the worksheet.)

Exercise 18. Perform the following elementary row operations, in order, on the matrix A.

1. Add -2 times row 1 to row 2.
2. Switch row 2 and row 3.

3. Add -1 times row 2 to row 1.
4. Multiply row 3 by $\frac{1}{3}$.

If you did the operations correctly, you will have row reduced A to I.

Exercise 19. Write down the four elementary matrices that correspond to the four row operations in the previous exercise. Call the first matrix (the matrix corresponding to adding -2 times row 1 to row 2) E_1, call the second matrix E_2, and so on.

Based on the discussion up to this point, you should be able to immediately write down E_1A without doing the multiplication. (If you aren't sure how to do this, multiply them together and see what happens). You should also be able to write down $E_2(E_1A)$. (Again, if you aren't sure, do the multiplication). Similarly for $E_3(E_2(E_1A))$ and $E_4(E_3(E_2(E_1A)))$. (Hint: $E_4(E_3(E_2(E_1A))) = I$.)

We're almost ready to write A as a product of elementary matrices. First, we need the inverses of E_1, E_2, E_3, and E_4.

Exercise 20. Write down E_1^{-1}, E_2^{-1}, E_3^{-1}, and E_4^{-1}.

We'll make use of the equation

$$E_4E_3E_2E_1A = I$$

First we'll multiply both sides of this equation *on the left* by E_4^{-1} (remember, matrix multiplication is not commutative, so we must be careful about whether we multiply on the right or left).

$$E_4^{-1}(E_4E_3E_2E_1A) = E_4^{-1}I$$

Now we'll do some simplification. On the left-hand side of the equation, we get

$$E_4^{-1}(E_4E_3E_2E_1A) = (E_4^{-1}E_4)E_3E_2E_1A = IE_3E_2E_1A = E_3E_2E_1A$$

On the right-hand side of the equation, we get

$$E_4^{-1}I = E_4^{-1}$$

Therefore we get a new equation

$$E_3E_2E_1A = E_4^{-1}$$

A few matrix multiplication rules were used. If you have time, read through the list in section 2.9 of your textbook and try to spot which rules were used.

Exercise 21. Continue this process by multiplying both sides of the equation $E_3E_2E_1A = E_4^{-1}$ on the left by E_3^{-1}, then by E_2^{-1}, and finally by E_1^{-1}. Simplify.

You should now have an equation that gives A as a product of elementary matrices.

It should be noted that there is more than way to write A as a product of elementary matrices. This is because there is more than one sequence of elementary row operations that will give you the reduced row echelon form of A.

Here is one more exercise.

Exercise 22. Verify that the matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ is invertible by showing that its reduced row echelon form is I. Then write B as a product of elementary matrices.