MATH 304

Test IV: The Voyage $\mathrm{Hom}(V, W)$

8/5/16

Name: Solutions

INSTRUCTIONS

Before the test begins, you must put away all electronic devices and any paper or written material. Your cell phone must be turned off. When instructed, turn over this cover page and begin the exam. Show all of your work and simplify your answers. You will have one hour to complete the exam.

Problem	1	2	3	4	5	6	TOTAL	Bonus
Maximum Score	28	20	50	16	16	20	150	5
Your Score								

1. (28 points) If a statement is true, circle **T**. If it is false, circle **F**.

(a) **T** **F** $\det(A) + \det(A^T) = 2\det(A)$

(b) **T** **F** If P_3 is the vector space of polynomials of degree less than or equal to three, and X is a set of polynomials which does not contain a polynomial of degree 2, then X does not span P_3.

(c) **T** **F** If A and B are invertible $n \times n$ matrices then

$$\det(B^{-1}A) = \frac{\det A}{\det B}$$

(d) **T** **F** If $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis of V and $\mathbf{x}$ is a vector in W, then there is a linear transformation $T\colon V \to W$ with the property that $T(\mathbf{v}_1) = \mathbf{x}$ and $T(\mathbf{v}_2) = -\mathbf{x}$.

(e) **T** **F** If A is an $n \times n$ matrix and $\det A = \pi$, then A is invertible.

(f) **T** **F** If A is an $n \times n$ matrix, then the linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is an isomorphism.

(g) **T** **F** If A is an 5×5 matrix, and the sum of the first three columns of A is equal to the fourth column (as vectors in $\mathbf{R}^n$), then $\det A = 0$.

2. (20 points) Fill in the blanks in the following statement.

Let V be a vector space with basis $X = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and W be a vector space with basis $Y = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$. Let F be a linear transformation from V to W. Then the matrix of F with respect to the bases X and Y, ${}_YF_X$, is the __m__ $\times$ __n__ matrix whose i-th column is the coordinate vector of __$F(\vec{v}_i)$__ with respect to the basis __Y__.

3. (50 points) Let V be the vector space of functions with basis

$$X = (\sin^2(x), \cos^2(x), \sin(x)\cos(x))$$

(a) Note that differentiation is a linear transformation from V to itself. Find the matrix ${}_X D_X$, where $D = \frac{d}{dx}$ is the derivative.

$${}_X D_X = \left[K_X D(\sin^2(x)) \quad K_X D(\cos^2(x)) \quad K_X D(\sin(x)\cos(x)) \right]$$

$$= \left[K_X(2\sin(x)\cos(x)) \quad K_X(-2\sin(x)\cos(x)) \quad K_X(-\sin^2(x) + \cos^2(x)) \right]$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & -2 & 0 \end{bmatrix}$$

(b) What is the dimension of the image of D?

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & -2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of ${}_X D_X$ is 2, so the image of D has dimension 2.

(c) What is the dimension of the kernel of D?

1.

(d) Another basis for V is $Y = (1, \sin(2x), \cos(2x))$. Find the change of basis matrix ${}_X I_Y$.

$$ {}_X I_Y = \left[K_X(1) \quad K_X(\sin(2x)) \quad K_X(\cos(2x)) \right] $$

$$ = \left[K_X(\sin^2(x) + \cos^2(x)) \quad K_X(2\sin(x)\cos(x)) \quad K_X(-\sin^2(x) + \cos^2(x)) \right] $$

$$ = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} $$

(e) What is ${}_Y I_X$? Give your answer in terms of ${}_X I_Y$.

$$ {}_Y I_X = ({}_X I_Y)^{-1} $$

(f) What is ${}_Y D_Y$? Give your answer in terms of the other matrices you've found.

$$ {}_Y D_Y = ({}_Y I_X)({}_X D_X)({}_X I_Y) $$

4. (16 points) Find the determinant of $M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Simplify your answer.

$$M \xrightarrow[R_4-R_1]{\substack{R_2-R_1 \\ R_3-R_1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[R_4-R_2]{R_3-R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_4-R_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M'$$

$\det M' = 1 \qquad \det M = (\det M')(1)^6 = 1$

5. (16 points) Find the determinant of $N = \begin{bmatrix} -3 & 0 & 5 & 0 & 0 \\ 2 & 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ -4 & 0 & 5 & 0 & 0 \\ 3 & 0 & 2 & 0 & -2 \end{bmatrix}$. Simplify your answer.

$$\det N = (-1)^{2+2}(4)\det\begin{bmatrix} -3 & 5 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ -4 & 5 & 0 & 0 \\ 3 & 2 & 0 & -2 \end{bmatrix}$$

$$= 4(-1)^{2+3}(3)\det\begin{bmatrix} -3 & 5 & 0 \\ -4 & 5 & 0 \\ 3 & 2 & -2 \end{bmatrix}$$

$$= -12(-1)^{3+3}(-2)\det\begin{bmatrix} -3 & 5 \\ -4 & 5 \end{bmatrix}$$

$$= 24\left((-3)(5)-(5)(-4)\right) = 24(-15+20)$$

$$= 120$$

6. (20 points) Here's a fun fact. The set of all linear transformations from a vector space V to a vector space W can be made into a vector space! This vector space is denoted by $\mathrm{Hom}(V, W)$. (Hom is short for homomorphism, which is a fancy math word). For example, $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m)$ is the vector space of $m \times n$ matrices, together with the usual matrix addition and scalar multiplication.

   The function defined by $A \mapsto \det A$ is a function from $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ to $\mathbf{R}^1$. We will denote this function by

   $$\det : \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}^1$$

   You may be wondering if this function is a linear transformation for a given n.

   (a) Is $\det : \mathrm{Hom}(\mathbf{R}^1, \mathbf{R}^1) \to \mathbf{R}^1$ a linear transformation? Justify your answer.

   $\det([a]+[b]) = \det([a+b]) = a+b = \det([a])+\det([b])$

   $\det(c[a]) = \det([ca]) = ca = c(\det([a]))$

   Yes, it is a linear transformation.

   (b) Is $\det : \mathrm{Hom}(\mathbf{R}^2, \mathbf{R}^2) \to \mathbf{R}^1$ a linear transformation? Justify your answer.

   Let $A \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$. A is a 2×2 matrix.

   cA can be obtained by multiplying both rows of A by c. According to a property of the determinant,

   $\det(cA) = c^2(\det A) \neq c(\det A)$.

   No, it is not a linear transformation.

Bonus. (5 points) Draw a picture in which the roles of animals and humans are reversed.