MATH 304

Test III: The Search for Span

7/29/16

Name: Solutions

INSTRUCTIONS

Before the test begins, you must put away all electronic devices and any paper or written material. Your cell phone must be turned off. When instructed, turn over this cover page and begin the exam. Show all of your work and simplify your answers. You will have one hour to complete the exam.

Problem	1	2	3	4	5	6	7	TOTAL	Bonus
Maximum Score	21	15	24	20	40	15	15	150	5
Your Score									

1. (21 points) If a statement is true, circle **T**. If it is false, circle **F**.

(a) **T** **F** The subset of $\mathbf{R}^3$ defined by the equation $x + 2y + 3z = 4$ is a subspace of $\mathbf{R}^3$.

(b) **T** **F** A collection of 4 vectors cannot span $\mathbf{R}^3$.

(c) **T** **F** The largest possible dimension of the column space of an 8×5 matrix is 3.

(d) **T** **F** If T is a linear transformation from $\mathbf{R}^5$ to $\mathbf{R}^4$, then the kernel of T has dimension at least 1.

(e) **T** **F** If V and W are subspaces of $\mathbf{R}^3$, then $V \cup W$ is a subspace of $\mathbf{R}^3$.

(f) **T** **F** If T is a linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^4$ and T is onto, then T is also one-to-one.

(g) **T** **F** If A is an $m \times n$ matrix and $\mathbf{b}$ is any vector in $\mathbf{R}^m$, then the set of vectors $\mathbf{x} \in \mathbf{R}^n$ which are solutions to $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathbf{R}^n$.

2. (15 points) Fill in the blanks in the following result from the text: If X is a subset of a vector space V, then X is a vector space if it satisfies the following 3 properties:

- $\mathbf{0} \in$ X.
- If $\mathbf{u}, \mathbf{v} \in$ X, then $\vec{u} + \vec{v} \in X$.
- If $c \in \mathbf{R}$ and $\mathbf{v} \in$ X, then $c\vec{v} \in X$.

3. (24 points) $X = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -3 \end{bmatrix} \right\}$.

(a) Which of the standard basis vectors in $\mathbb{R}^3$ are in the span of X?

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 3 & -2 & 4 & 0 & 1 & 0 \\ -1 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-3R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -5 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Only $\vec{e_2}$ is in Span(X).

(b) Is X linearly independent?

No.

4. (20 points) Let $X = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. X is a linearly independent subset of $\mathbb{R}^4$, and hence it is a basis for $V = \mathrm{Span}(X)$. Verify that the vector $\mathbf{u} = \begin{bmatrix} 4 \\ -2 \\ 1 \\ -10 \end{bmatrix}$ is in V and evaluate $K_X(\mathbf{u})$, where K_X is the coordinate transformation of V with respect to the basis X.

$$\left[\begin{array}{ccc|c} 1 & -4 & -1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ -2 & 8 & 0 & -10 \end{array}\right] \xrightarrow{R_4+2R_1} \left[\begin{array}{ccc|c} 1 & -4 & -1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{R_1+4R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right] \xrightarrow[R_4+2R_3]{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$K_X(\vec{u}) = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

5. (40 points) Let $A = \begin{bmatrix} 1 & 2 & -4 & 4 & 3 \\ 0 & 3 & -6 & 8 & 1 \\ 1 & 1 & -2 & 1 & 3 \\ 1 & -1 & 2 & -4 & 2 \end{bmatrix}$. A may be viewed as a linear transformation $A : \mathbb{R}^5 \to \mathbb{R}^4$. The row reduced echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Write a basis for the kernel of A.

$$\begin{aligned} x_1 &= -x_5 \\ x_2 &= 2x_3 - 3x_5 \\ x_3 &= x_3 \\ x_4 &= x_5 \\ x_5 &= x_5 \end{aligned}$$

$$\vec{x} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

A basis for the kernel is $\left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$

(b) Write a basis for the image of A.

$$\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 1 \\ -4 \end{bmatrix} \right)$$

(c) Write a basis for the row space of A.

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \end{bmatrix} \right)$$

(d) Let $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}$. Write the solution set of $A\mathbf{x} = \mathbf{b}$ in vector parametric form. To get you started, the row reduced echelon form of $[A|\mathbf{b}]$ is $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

All solutions are of the form $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

6. (15 points) Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$. Find a basis for $\mathbb{R}^3$ that contains S.

$$\left[\begin{array}{cc:c} 1 & -1 & a \\ 2 & 1 & b \\ 0 & 2 & c \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc:c} 1 & -1 & a \\ 0 & 1 & b-2a \\ 0 & 2 & c \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{cc:c} 1 & -1 & a \\ 0 & 1 & b-2a \\ 0 & 0 & c-2b+4a \end{array}\right]$$

Any vector $\begin{bmatrix} a \\ b-2a \\ c-2b+4a \end{bmatrix}$ such that $c-2b+4a \neq 0$ will work. For example, let $a=1$, $b=0$, and $c=0$.

Then $\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right)$ is a basis for $\mathbb{R}^3$.

7. (15 points) Prove that $\{1, e^x\}$ is a linearly independent subset of the vector space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$.

1 is not a scalar multiple of e^x:

$$1 \neq ce^x \text{ for any } c$$

e^x is not a scalar multiple of 1:

$$e^x \neq c(1) \text{ for any } c$$

Therefore $\{1, e^x\}$ is linearly independent.

Bonus. (5 points) Write a poem about linear algebra.