MATH 304

Test II: The Math of Khan

7/22/16

Name: Solutions

INSTRUCTIONS

Before the test begins, you must put away all electronic devices and any paper or written material. Your cell phone must be turned off. When instructed, turn over this cover page and begin the exam. Show all of your work and simplify your answers. You will have one hour to complete the exam.

Problem	1	2	3	4	5	6	7	8	9	TOTAL	Bonus
Maximum Score	8	10	12	15	20	25	20	25	15	150	5
Your Score											

1. (8 points) Complete the list of 8 axioms for a vector space. $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors, and c_1, c_2, c, and 1 are scalars.

   (a) $\mathbf{x}+\mathbf{y} =$ $\vec{y} + \vec{x}$
   (b) $(\mathbf{x}+\mathbf{y})+\mathbf{z} =$ $\vec{x} + (\vec{y} + \vec{z})$
   (c) $\mathbf{x}+\mathbf{0} =$ $\vec{x}$
   (d) $\mathbf{x}+(-\mathbf{x}) =$ $\vec{0}$
   (e) $c_1(c_2\mathbf{x}) =$ $(c_1 c_2)\vec{x}$
   (f) $c(\mathbf{x}+\mathbf{y}) =$ $c\vec{x} + c\vec{y}$
   (g) $(c_1+c_2)\mathbf{x} =$ $c_1\vec{x} + c_2\vec{x}$
   (h) $1\mathbf{x} =$ $\vec{x}$

2. (10 points) If a statement is true, circle **T**. If it is false, circle **F**.

   (a) **T** **F** The set of all $m \times n$ matrices is a vector space.
   (b) **T** **F** Every $m \times n$ matrix is a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$.
   (c) **T** **F** $(A+B)C = AC+BC$ (provided the products and sums are defined).
   (d) **T** **F** $(AB)C = A(BC)$ (provided the products are defined).
   (e) **T** **F** If M and N are invertible $n \times n$ matrices, then $(MN)^{-1} = N^{-1}M^{-1}$.

3. (12 points) Let $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

   (a) Is the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^3$ onto? Is it one-to-one?

   No, it is not onto.
   Yes, it is one-to-one.

   (b) Is A invertible?

   No.

4. (15 points) Fill in the blanks in the statement of the following theorems.

   (a) For a given matrix A, there is only one matrix in reduced row echelon form that is row equivalent to A.

   (b) Let C be an $m \times n$ matrix regarded as a function from $\mathbf{R}^n$ to $\mathbf{R}^m$. Let r be the rank of C. Then C is onto if and only if $r = m$, and C is one-to-one if and only if $r = n$.

5. (20 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by the matrix

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

and let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix

$$\begin{bmatrix} 0 & 2 & -2 \\ 3 & 0 & -2 \\ 3 & -1 & 0 \end{bmatrix}$$

The composition ST is also linear transformation. Find the unique matrix C such that $ST(\mathbf{x}) = C\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

$$C = \begin{bmatrix} 0 & 2 & -2 \\ 3 & 0 & -2 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & -4 \\ 3 & -6 \end{bmatrix}$$

6. (25 points)

(a) Find all possible solutions to the following matrix equation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix} X = \begin{bmatrix} 2 & -2 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 0 \\ 1 & 2 & 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 0 \\ 0 & 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \end{array}\right]$$

The only solution is $X = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$

(b) Without doing any row reduction, find all possible solutions to the following matrix equation. (Hint: you will need the correct solution to the previous problem, along with some facts about the transpose of a matrix.)

$$\begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} Y = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

Let $M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Then

part (a) shows that $MX = B$. Therefore

$$X^T M^T = (MX)^T = B^T$$

This shows that $M^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ is a solution

to the equation.

7. (20 points)

(a) Consider the function T from $\mathbf{R}^2$ to $\mathbf{R}^2$ given by rotation of the plane counter-clockwise by 90 degrees around the origin.

i. Is T a linear transformation? Yes

ii. If your answer in the previous part was yes, write the standard matrix of T below. If your answer was no, explain why T is not a linear transformation.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(b) Consider the function S from $\mathbf{R}^2$ to $\mathbf{R}^2$ given by $S(x_1, x_2) = (2x_2, 1)$

i. Is S a linear transformation? No

ii. If your answer in the previous part was yes, write the standard matrix of S below. If your answer was no, explain why T is not a linear transformation.

$$S(\vec{x}+\vec{y}) = S((x_1, x_2) + (y_1, y_2)) = S(x_1+y_1, x_2+y_2)$$
$$= (2(x_2+y_2), 1)$$

$$S(\vec{x}) + S(\vec{y}) = S(x_1, x_2) + S(y_1, y_2)$$
$$= (2x_2, 1) + (2y_2, 1) = (2x_2+2y_2, 2)$$

$S(\vec{x}+\vec{y}) \neq S(\vec{x}) + S(\vec{y})$, so S is not a linear transformation.

8. (25 points) Let $A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 9 \end{bmatrix}$

(a) Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{9} \end{array}\right] \xrightarrow{R_1+2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & \frac{2}{9} \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{9} \end{array}\right]$$

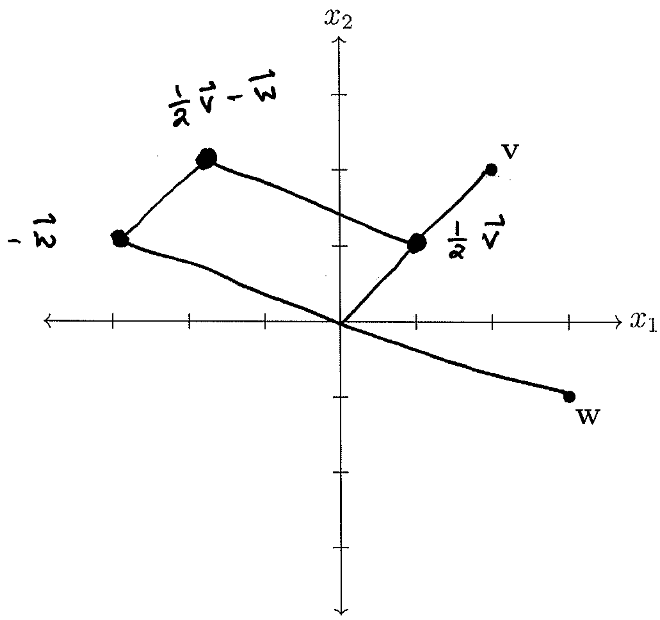
$$\xrightarrow{R_2+2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & \frac{2}{9} \\ 0 & 1 & 0 & 0 & 1 & \frac{2}{9} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{9} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -2 & \frac{2}{9} \\ 0 & 1 & \frac{2}{9} \\ 0 & 0 & \frac{1}{9} \end{bmatrix}$$

(b) Write A as a product of elementary matrices.

$$A = \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\Updownarrow \atop R_1-2R_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{\Updownarrow \atop 9R_2} \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\Updownarrow \atop R_1-2R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{\Updownarrow \atop R_2-2R_3}$$

9. (15 points) On the graph below, draw the vector $\frac{1}{2}\mathbf{v} - \mathbf{w}$.



Bonus. (5 points) Write down exactly one integer between 0 and 100 (possibly 0 or 100). 5 points will be awarded for writing the largest unique integer, 4 points for the next largest unique integer, and so on. If you choose the same integer as someone else then that integer is not unique and you won't get any bonus points.