# MATH 324
## Exam 2
## Solutions

1/6/17

## INSTRUCTIONS

Before the test begins, you must put away all electronic devices. You may use a copy of the tables from your textbook, but no other notes are allowed. Your cell phone must be turned off. When instructed, turn over this cover page and begin the exam. Show all of your work and simplify your answers. You will have two hours to complete the exam.

| Problem | 1 | 2 | 3 | 4 | 5 | TOTAL | Bonus |
|---|---|---|---|---|---|---|---|
| Maximum Score | 25 | 25 | 25 | 25 | 25 | 125 | 10 |
| Your Score | | | | | | | |

1. (25 points) Use an appropriate infinite series method about $x = 0$ to find two solutions of the given differential equation.

$$9x^2y'' + x^2y' + 2y = 0$$

0 is a regular singular point, so we will look for solutions of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

$$9x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} + x^2 \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} + 2 \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} 9c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r+1} + \sum_{n=0}^{\infty} 2c_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} 9c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=1}^{\infty} c_{n-1} (n+r-1) x^{n+r} + \sum_{n=0}^{\infty} 2c_n x^{n+r}$$

$$= x^r \left( 9r(r-1)c_0 + 2c_0 + \sum_{n=1}^{\infty} (9c_n(n+r)(n+r-1) + c_{n-1}(n+r-1) + 2c_n) x^n \right)$$

The indicial equation is $9r(r-1) + 2 = 0$

$$9r^2 - 9r + 2 = 0$$

$$(3r-2)(3r-1) = 0$$

$$r_1 = \frac{2}{3}, \quad r_2 = \frac{1}{3}$$

$$9c_n(n+r)(n+r-1) + c_{n-1}(n+r-1) + 2c_n = 0$$

$$c_n = \frac{-c_{n-1}(n+r-1)}{(n+r)(n+r-1)+2} \text{ when } n \geq 1$$

$$\text{If } r_1 = \frac{2}{3}, \text{ then } c_n = \frac{-c_{n-1}(n+\frac{2}{3}-1)}{(n+\frac{2}{3})(n+\frac{2}{3}-1)+2} = \frac{-3c_{n-1}(3n-1)}{(3n+2)(3n-1)+18}$$

$$c_0 = \text{ anything}, \quad c_1 = -\frac{3}{14}c_0, \quad c_2 = -\frac{15}{58}c_1 = \frac{3 \cdot 15}{14 \cdot 58}c_0, \quad c_3 = -\frac{12}{53}c_2 = -\frac{3 \cdot 15 \cdot 12}{14 \cdot 58 \cdot 53}c_0$$

$$y_1 = x^{\frac{2}{3}} \left( 1 - \frac{3}{14}x + \frac{3 \cdot 15}{14 \cdot 58}x^2 - \frac{3 \cdot 15 \cdot 12}{14 \cdot 58 \cdot 53}x^3 + \dots \right)$$

$$\text{If } r_2 = \frac{1}{3}, \text{ then } c_n = \frac{-c_{n-1}(n+\frac{1}{3}-1)}{(n+\frac{1}{3})(n+\frac{1}{3}-1)+2} = \frac{-3c_{n-1}(3n-2)}{(3n+1)(3n-2)+18}$$

$$c_0 = \text{ anything}, \quad c_1 = -\frac{3}{22}c_0, \quad c_2 = -\frac{6}{23}c_1 = \frac{3 \cdot 6}{22 \cdot 23}c_0, \quad c_3 = -\frac{21}{88}c_2 = -\frac{3 \cdot 6 \cdot 21}{22 \cdot 23 \cdot 88}c_0$$

$$y_2 = x^{\frac{1}{3}} \left( 1 - \frac{3}{22}x + \frac{3 \cdot 6}{22 \cdot 23}x^2 - \frac{3 \cdot 6 \cdot 21}{22 \cdot 23 \cdot 88}x^3 + \dots \right)$$

2. (25 points) Find the general solution of the differential equation.

$$y'' + 2y' + y = (x-1)^2$$

To find $y_c$, we'll look for solutions of the form $y = e^{mx}$. To find $y_p$, we'll use the method of undetermined coefficients.

$$m^2 + 2m + 1 = 0$$
$$(m+1)^2 = 0$$
$$y_c = c_1 e^x + c_2 x e^x$$
$$y_p = Ax^2 + Bx + C$$
$$y_p' = 2Ax + B$$
$$y_p'' = 2A$$
$$2A + 2(2Ax + B) + Ax^2 + Bx + C = x^2 - 2x + 1$$
$$Ax^2 + (4A + B)x + 2A + 2B + C = x^2 - 2x + 1$$
$$A = 1$$
$$4A + B = -2$$
$$2A + 2B + C = 1$$
$$B = -6, \quad C = -11$$
$$y_p = x^2 - 6x - 11$$
$$y = y_c + y_p$$

3. (25 points) Use an appropriate infinite series method about $x = 0$ to find two solutions of the given differential equation.

$$y'' + xy' + y = 0$$

0 is an ordinary point, so we will look for solutions of the form $y = \sum_{n=0}^{\infty} c_n x^n$.

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n n x^{n} + \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^{n} + \sum_{n=1}^{\infty} c_n n x^{n} + \sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2 + c_0 + \sum_{n=1}^{\infty}(c_{n+2}(n+2)(n+1) + c_n n + c_n)x^n$$

$$= 2c_2 + c_0 + \sum_{n=1}^{\infty}(c_{n+2}(n+2)(n+1) + (n+1)c_n)x^n$$

$$2c_2 + c_0 = 0$$

$$c_{n+2} = \frac{-c_n(n+1)}{(n+2)(n+1)} = -\frac{c_n}{(n+2)} \text{ when } n \geq 1$$

$$c_0 = \text{ anything}, \quad c_1 = \text{ anything}$$

$$c_2 = -\frac{1}{2}c_0, \quad c_3 = -\frac{1}{3}c_1$$

$$c_4 = -\frac{1}{4}c_2 = \frac{1}{2\cdot 4}c_0, \quad c_5 = -\frac{1}{5}c_3 = \frac{1}{3\cdot 5}c_1$$

$$c_6 = -\frac{1}{6}c_4 = -\frac{1}{2\cdot 4\cdot 6}c_0, \quad c_7 = -\frac{1}{7}c_5 = -\frac{1}{3\cdot 5\cdot 7}c_1$$

$$y_1 = 1 - \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 - \frac{1}{2\cdot 4\cdot 6}x^6 + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n}$$

$$y_2 = x - \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 - \frac{1}{3\cdot 5\cdot 7}x^7 + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot \cdots \cdot (2n+1)}x^{2n+1}$$

4. (25 points) Find the general solution of the differential equation.

$$2y'' + 2y' + y = x\csc(x)$$

To find $y_c$, we'll look for solutions of the form $y = e^{mx}$. To find $y_p$, we'll use variation of parameters.

$$2m^2 + 2m + 1 = 0$$

$$m = \frac{-2 \pm \sqrt{4-8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$$

$$y_1 = e^{-\frac{1}{2}x}\cos\left(\frac{1}{2}x\right)$$

$$y_2 = e^{-\frac{1}{2}x}\sin\left(\frac{1}{2}x\right)$$

$$y_c = c_1y_1 + c_2y_2$$

$$y_1' = -\frac{1}{2}e^{-\frac{1}{2}x}\cos\left(\frac{1}{2}x\right) - \frac{1}{2}e^{-\frac{1}{2}x}\sin\left(\frac{1}{2}x\right)$$

$$y_2' = -\frac{1}{2}e^{-\frac{1}{2}x}\sin\left(\frac{1}{2}x\right) + \frac{1}{2}e^{-\frac{1}{2}x}\cos\left(\frac{1}{2}x\right)$$

$$y_c = c_1y_1 + c_2y_2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' = \frac{1}{2}e^{-\frac{1}{2}x}\cos^2\left(\frac{1}{2}x\right) + \frac{1}{2}e^{-\frac{1}{2}x}\sin^2\left(\frac{1}{2}x\right) = \frac{1}{2}e^{-\frac{1}{2}x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ \frac{x\csc(x)}{2} & y_2' \end{vmatrix} = -e^{-\frac{1}{2}x}\sin\left(\frac{1}{2}x\right)\frac{x\csc(x)}{2}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & \frac{x\csc(x)}{2} \end{vmatrix} = e^{-\frac{1}{2}x}\cos\left(\frac{1}{2}x\right)\frac{x\csc(x)}{2}$$

$$u_1' = \frac{W_1}{W} = -\sin\left(\frac{1}{2}x\right)x\csc(x)$$

$$u_2' = \frac{W_2}{W} = \cos\left(\frac{1}{2}x\right)x\csc(x)$$

$$u_1 = \int \frac{W_1}{W}dx = \int_{x_0}^{x} -\sin\left(\frac{1}{2}t\right)t\csc(t)dt$$

$$u_2 = \int \frac{W_2}{W}dx = \int_{x_0}^{x} \cos\left(\frac{1}{2}t\right)t\csc(t)dt$$

$$y_p = u_1y_1 + u_2y_2$$

$$y = y_c + y_p$$

5. (25 points) Find the general solution of the differential equation.

$$4x^2y'' + 4xy' + y = 0$$

This is a Cauchy-Euler equation, so we'll look for solutions of the form $y = x^m$.

$$4m(m-1) + 4m + 1 = 0$$
$$4m^2 + 1 = 0$$
$$m = \pm\frac{1}{2}i$$
$$y = c_1 \cos\left(\frac{1}{2}\ln(x)\right) + c_2 \sin\left(\frac{1}{2}\ln(x)\right)$$

Bonus. (5 points) Choose a positive integer. Five points will be awarded to anyone who chooses the smallest unique integer. The second-smallest unique integer is worth four points, the third-smallest unique integer is worth three points, and so on. Anyone who chooses the same integer as someone else in the class will receive no extra points.

Your positive integer: ___________

Bonus. (5 points, no partial credit) Find a non-trivial solution to the non-linear differential equation.

$$6\frac{d^2y}{dx^2} = y^2$$

Try to find a series solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$.