MATH 324
Exam 1
Solutions

12/23/16

INSTRUCTIONS

Before the test begins, you must put away all electronic devices. You may use a copy of the tables from your textbook, but no other notes are allowed. Your cell phone must be turned off. When instructed, turn over this cover page and begin the exam. Show all of your work and simplify your answers. You will have two hours to complete the exam.

Problem	1	2	3	4	5	6	TOTAL	Bonus
Maximum Score	20	20	20	20	20	20	120	10
Your Score								

1. (20 points) Solve the given initial-value problem.

$$y'' - 4y' - 5y = 0, \quad y(0) = 5, \quad y'(0) = 7$$

This is a homogeneous equation with constant coefficients, so we will look for solutions of the form $y = e^{mx}$.

$$m^2 - 4m - 5 = 0$$
$$(m-5)(m+1) = 0$$
$$m_1 = 5$$
$$m_2 = -1$$
$$y = c_1 e^{5x} + c_2 e^{-x}$$
$$y' = 5c_1 e^{5x} - c_2 e^{-x}$$
$$y(0) = c_1 + c_2 = 5$$
$$y'(0) = 5c_1 - c_2 = 7$$
$$c_1 = 2$$
$$c_2 = 3$$
$$y = 2e^{5x} + 3e^{-x}$$

2. (20 points) Solve the given differential equation.

$$xdy - 4(y + x^6)dx = 0$$

We will put the equation in standard form and and observe that it is linear. Then we'll use an integrating factor to solve the equation.

$$x\frac{dy}{dx} - 4y - 4x^6 = 0$$
$$\frac{dy}{dx} - \frac{4}{x}y = 4x^5$$
$$\int P(x) = \int -\frac{4}{x}dx = -4\ln(x) = \ln\left(\frac{1}{x^4}\right)$$
$$\mu(x) = e^{\int P(x)} = e^{\ln\left(\frac{1}{x^4}\right)} = \frac{1}{x^4}$$
$$\mu(x)\frac{dy}{dx} - \mu(x)\frac{4}{x}y = \mu(x)4x^5$$
$$\frac{d}{dx}(\mu(x)y) = \mu(x)4x^5$$
$$\frac{d}{dx}\left(\frac{1}{x^4}y\right) = 4x$$
$$\int \frac{d}{dx}\left(\frac{1}{x^4}y\right)dx = \int 4xdx$$
$$\frac{1}{x^4}y = 2x^2 + C$$
$$y = 2x^6 + Cx^4$$

It should be noted that $P(x)$ has an infinite discontinuity at 0. Therefore this is the most general solution only on the interval $(0, \infty)$.

3. (20 points) Solve the given differential equation.

$$(x - y^3 + y^2)dx = (3xy^2 - 2yx)dy$$

First we check to see if the equation is exact. It is exact, so we can solve it directly.

$$(x - y^3 + y^2)dx + (-3xy^2 + 2yx)dy = 0$$
$$M(x, y) = x - y^3 + y^2$$
$$N(x, y) = -3xy^2 + 2yx$$
$$M_y = -3y^2 + 2y$$
$$N_x = -3y^2 + 2y$$
$$M_y = N_x, \text{ so the equation is exact.}$$
$$f(x, y) = \int M(x, y)dx = \int x - y^3 + y^2 dx = \frac{x^2}{2} - xy^3 + xy^2 + g(y)$$
$$\frac{\partial f}{\partial y} = -3xy^2 + 2xy + g'(y) = N(x, y) = -3xy^2 + 2yx$$
$$g'(y) = 0$$
$$g(y) = C$$
$$f(x, y) = \frac{x^2}{2} - xy^3 + xy^2 + C$$
$$\frac{x^2}{2} - xy^3 + xy^2 + C = 0$$

4. (20 points) Solve the given differential equation.

$$\frac{dy}{dx} = \frac{y^2+1}{x^2+4}$$

This equation is separable.

$$\frac{dy}{dx} = \frac{y^2+1}{x^2+4}$$
$$\frac{dy}{y^2+1} = \frac{dx}{x^2+4}$$
$$\int \frac{dy}{y^2+1} = \int \frac{dx}{x^2+4}$$
$$\arctan(y) = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$
$$y = \tan\left(\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C\right)$$

5. (20 points) Solve the given differential equation.

$$y\frac{dy}{dx} + y^2 = 1$$

This equation is separable.

$$\frac{dy}{dx} = \frac{1}{y} - y = \frac{1-y^2}{y}$$
$$\frac{1-y^2}{dy} = dx$$
$$\int \frac{1-y^2}{dy} = \int dx$$
$$-\frac{1}{2}\ln(1-y^2) = x + C_1$$
$$\ln\left(\frac{1}{\sqrt{1-y^2}}\right) = x + C_1$$
$$\frac{1}{\sqrt{1-y^2}} = C_2 e^x$$
$$1 - y^2 = Ce^{-2x}$$
$$y^2 = 1 - Ce^{-2x}$$

6. (20 points) Find the general solution of the differential equation.

$$4\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + y = 0$$

This is a homogeneous equation with constant coefficients, so we will look for solutions of the form $y = e^{mx}$.

$$m^4 + 4m^2 + 1 = 0$$

$$m^2 = \frac{-4 \pm \sqrt{48}}{8} = -1 \pm \frac{\sqrt{3}}{2}$$

$$m = \pm\sqrt{-1 \pm \frac{\sqrt{3}}{2}}$$

$$m_1 = \sqrt{-1 + \frac{\sqrt{3}}{2}}$$

$$m_2 = -\sqrt{-1 + \frac{\sqrt{3}}{2}}$$

$$m_3 = \sqrt{-1 - \frac{\sqrt{3}}{2}} = i\sqrt{1 + \frac{\sqrt{3}}{2}}$$

$$m_4 = -\sqrt{-1 - \frac{\sqrt{3}}{2}} = -i\sqrt{1 + \frac{\sqrt{3}}{2}}$$

$$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3\sin\left(\sqrt{1 + \frac{\sqrt{3}}{2}}x\right) + c_4\sin\left(\sqrt{1 + \frac{\sqrt{3}}{2}}x\right)$$

Bonus. (5 points) Choose a prime number and write it below. Five points will be awarded to anyone who chooses the second-largest prime number out of the set of primes chosen by everyone in the class. The third-largest prime is worth four points, the fourth-largest prime is worth three points, and so on. Anyone who chooses the largest prime will receive no extra points.

Bonus. (5 points) Solve the given differential equation.

$$\left(x \ln\left(\frac{x}{y}\right) - y\right)\frac{dy}{dx} = y \ln\left(\frac{x}{y}\right)$$

Substitute $x = vy$.