Supplementary Exercises – Chapter 13 (answers below)

1. State the Mean Value Theorem.

2. Explain the difference between the Mean Value Theorem and Rolle's Theorem.

3. Verify the hypotheses of the Mean Value Theorem for each function below defined on the indicated interval. Then find the value "c" referred to by the theorem.

   (a) $f(x)=\sqrt{x+1}$ $[3,\ 8]$

   (b) $g(x)=\dfrac{x-1}{x+1}$ $[0,\ 3]$

   (c) $h(x)=\dfrac{x^2-2x-3}{x+4}$ $[-1,\ 3]$

   (d) $k(x)=x^{\frac{3}{4}}-2x^{\frac{1}{4}}$ $[0,\ 4]$ Hint: It will be easier to evaluate *k(4)* if you first factor *k*.

4. Which function(s) in Problem #3 is an example of Rolle's Theorem?

5. For each of the functions below, there is no "c" in (a, b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$. Explain why this is not a contradiction to the Mean Value Theorem.

   (a) $f(x)=\dfrac{3}{x-2}$ $[1,\ 3]$

   (b) $g(x)=\begin{cases} x^2+1 & 0\le x<1 \\ 3-x & 1\le x\le 5 \end{cases}$

6. Suppose the f is a function such that $f(0)=-3$ and $f'(x)\le 5$ for all x in $\mathfrak{R}$. What is the largest possible value for $f(2)$?

Answers:

1. If a function f is continuous on a closed interval $[a,\ b]$ and is differentiable on the open interval $(a,\ b)$ then there is at least one number c in the interval $(a,\ b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

2. Rolle's Theorem is a special case of the Mean Value Theorem. In addition to the two requirements (continuity and differentiability) of the Mean Value Theorem, Rolle's Theorem requires that $f(a)=f(b)$. The conclusion for Rolle's Theorem is the existence of some c in $(a,\ b)$ such that $f'(c)=0$.

3. For all of the problems, you need to show continuity on the closed interval and differentiability on the open interval. To show differentiability, simply take the derivative

and show that is defined on the open interval. To show continuity you can use the definition directly or you could use the fact that if f is differentiable at a point then it is also continuous at that point. An illustration of this is given below for problems (a) and (d).

(a) $f'(x) = \frac{1}{2\sqrt{x+1}}$. $f'(x)$ is defined for all values of x greater than -1. So, f is certainly differentiable on $(3, 8)$. Since differentiability implies continuity, f is continuous on interval $[3, 8]$.

(d) $k'(x) = \frac{3}{4}x^{-\frac{1}{4}} - \frac{1}{2}x^{-\frac{3}{4}}$. $k'(x)$ is defined for all $x > 0$, so k is differentiable on interval $(0, 4)$. This differentiability implies continuity on the interval $(0, 4]$ but does not give us continuity at $x = 0$. This we must do by the definition of continuity: $\lim_{x \to 0^+} k(x) = 0^{\frac{3}{4}} - 2(0)^{\frac{1}{4}} = k(0)$, so k is right-continuous at 0. Thus k is continuous on all of the closed interval $[0, 4]$.

(a) $c = \frac{21}{4}$

(b) $c = 1$ (note: $c = -2$ is not in the required interval, so it is not a number referred to by the Mean Value Theorem).

(c) $c = -4 + \sqrt{21}$ (note: $c = -4 - \sqrt{21}$ is not in the required interval, so it is not a number referred to by the Mean Value Theorem).

(d) $c = \frac{4}{9}$

4. Problem (c) is an example of Rolle's Theorem. $f(-1) = f(3)$.
   Also, problem (d) is an example of Rolle's Theorem. $f(0) = f(4)$.
   Note: In both problems (c) and (d) it was true that $f(a) = 0$ and $f(b) = 0$. This is not required of Rolles's Theorem. Rolle's Theorem only requires that $f(a) = f(b)$.

5. (a) f does not meet the requirements for the Mean Value Theorem. $f(2)$ does not exist, so f is discontinuous at $x = 2$.

   (b) g does not meet the requirements for the Mean Value Theorem. $g'(2)$ does not exist, so g is not differentiable on interval $(0, 5)$.

6. $f(2) \leq 7$