Section 3-7 Marginal Analysis in Business and Economics

- Marginal Cost, Revenue, and Profit
- Application
- Marginal Average Cost, Revenue, and Profit

Marginal Cost, Revenue, and Profit

One important use of calculus in business and economics is in *marginal analysis*. In economics, the word *marginal* refers to a rate of change, that is, to a derivative. Thus, if $C(x)$ is the total cost of producing x items, then $C'(x)$ is called the *marginal cost* and represents the instantaneous rate of change of total cost with respect to the number of items produced. Similarly, the *marginal revenue* is the derivative of the total revenue function and the *marginal profit* is the derivative of the total profit function.

DEFINITION Marginal Cost, Revenue, and Profit

If x is the number of units of a product produced in some time interval, then

$$\begin{aligned} \text{total cost} &= C(x) \\ \textbf{marginal cost} &= C'(x) \\ \text{total revenue} &= R(x) \\ \textbf{marginal revenue} &= R'(x) \\ \text{total profit} &= P(x) = R(x) - C(x) \\ \textbf{marginal profit} &= P'(x) = R'(x) - C'(x) \\ &= (\text{marginal revenue}) - (\text{marginal cost}) \end{aligned}$$

Marginal cost (or revenue or profit) is the instantaneous rate of change of cost (or revenue or profit) relative to production at a given production level.

To begin our discussion, we consider a cost function $C(x)$. It is important to remember that $C(x)$ represents the *total* cost of producing x items, not the cost of producing a *single* item. To find the cost of producing a single item, we use the difference of two successive values of $C(x)$:

$$\begin{aligned} \text{Total cost of producing } x + 1 \text{ items} &= C(x + 1) \\ \text{Total cost of producing } x \text{ items} &= C(x) \\ \text{Exact cost of producing the } (x + 1)\text{st item} &= C(x + 1) - C(x) \end{aligned}$$

You Try It

EXAMPLE 1 **Cost Analysis** A company manufactures fuel tanks for automobiles. The total weekly cost (in dollars) of producing x tanks is given by

$$C(x) = 10{,}000 + 90x - 0.05x^2$$

(A) Find the marginal cost function.

(B) Find the marginal cost at a production level of 500 tanks per week and interpret the results.

(C) Find the exact cost of producing the 501st item.

Solution

(A) $C'(x) = 90 - 0.1x$

(B) $C'(500) = 90 - 0.1(500) = \$40$ Marginal cost

At a production level of 500 tanks per week, the total production costs are increasing at the rate of $40 tank.

(C) $$\begin{aligned} C(501) &= 10{,}000 + 90(501) - 0.05(501)^2 \\ &= \$42{,}539.95 \quad \text{Total cost of producing 501 tanks per week} \\ C(500) &= 10{,}000 + 90(500) - 0.05(500)^2 \\ &= \$42{,}500.00 \quad \text{Total cost of producing 500 tanks per week} \\ C(501) - C(500) &= 42{,}539.95 - 42{,}500.00 \\ &= \$39.95 \quad \text{Exact cost of producing the 501st tank} \end{aligned}$$

Matched Problem 1 A company manufactures automatic transmissions for automobiles. The total weekly cost (in dollars) of producing x transmissions is given by

$$C(x) = 50{,}000 + 600x - 0.75x^2$$

(A) Find the marginal cost function.
(B) Find the marginal cost at a production level of 200 transmissions per week and interpret the results.
(C) Find the exact cost of producing the 201st transmission.

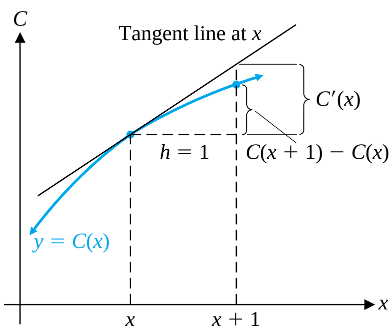

FIGURE 1 $C'(x) \approx C(x+1) - C(x)$

In Example 1, we found that the cost of the 501st tank and the marginal cost at a production level of 500 tanks differ by only a nickel. To explore this apparent relationship between marginal cost and cost of a single item, we return to the definition of the derivative. If $C(x)$ is any total cost function, then

$$C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h} \qquad \text{Marginal cost}$$

$$C'(x) \approx \frac{C(x+h) - C(x)}{h} \qquad h \neq 0$$

$$C'(x) \approx C(x+1) - C(x) \qquad h = 1$$

Thus, we see that the marginal cost $C'(x)$ approximates $C(x+1) - C(x)$, the exact cost of producing the $(x+1)$st item. These observations are summarized below and illustrated in Figure 1.

THEOREM 1 Marginal Cost and Exact Cost

If $C(x)$ is the total cost of producing x items, the marginal cost function approximates the exact cost of producing the $(x+1)$st item:

$$\overset{\text{Marginal cost}}{C'(x)} \approx \overset{\text{Exact cost}}{C(x+1) - C(x)}$$

Similar statements can be made for total revenue functions and total profit functions.

Insight Theorem 1 states that the marginal cost at a given production level x approximates the cost of producing the $(x+1)$st or *next* item. In practice, the marginal cost is used more frequently than the exact cost. One reason for this is that the marginal cost is easily visualized when examining the graph of the total cost function. Figure 2 shows the graph of the cost function discussed in Example 1 with tangent lines added at $x = 200$ and $x = 500$. The graph clearly shows that as production increases, the slope of the tangent line decreases. Thus, the cost of producing the next tank also decreases, a desirable characteristic of a total cost function. We will have much more to say about graphical analysis in Chapter 4.

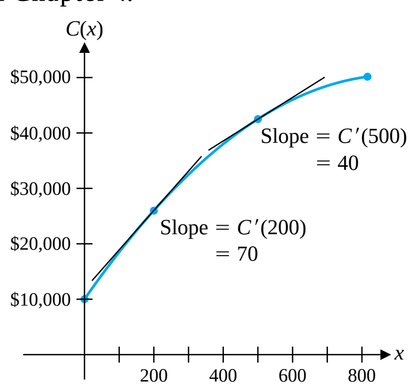

FIGURE 2 $C(x) = 10{,}000 + 90x - 0.05x^2$

➤ Application

We now want to discuss how price, demand, revenue, cost, and profit are tied together in typical applications. Although either price or demand can be used as the independent variable in a price–demand equation, it is common practice to use demand as the independent variable when marginal revenue, cost, and profit are also involved.

EXPLORE & DISCUSS 1

The market research department of a company used test marketing to determine the demand for a new radio (Table 1).

(A) Assuming that the relationship between price p and demand x is linear, find the price–demand equation and write the result in the form $x = f(p)$. Graph the equation and find the domain of f. Discuss the effect of price increases and decreases on demand.

(B) Solve the equation found in part (A) for p, obtaining an equation of the form $p = g(x)$. Graph this equation and find the domain of g. Discuss the effect of price increases and decreases on demand.

TABLE 1

Demand x	Price p
3,000	\$7
6,000	\$4

EXAMPLE 2

Production Strategy The market research department of a company recommends that the company manufacture and market a new transistor radio. After suitable test marketing, the research department presents the following **price–demand equation:**

$$x = 10{,}000 - 1{,}000p \quad x \text{ is demand at price } p. \tag{1}$$

Or, solving (1) for p,

$$p = 10 - 0.001x \tag{2}$$

where x is the number of radios retailers are likely to buy at \$$p$ per radio.

The financial department provides the following **cost function:**

$$C(x) = 7{,}000 + 2x \tag{3}$$

where \$7,000 is the estimate of fixed costs (tooling and overhead) and \$2 is the estimate of variable costs per radio (materials, labor, marketing, transportation, storage, etc.).

(A) Find the domain of the function defined by the price–demand equation (2).

(B) Find the marginal cost function $C'(x)$ and interpret.

(C) Find the revenue function as a function of x, and find its domain.

(D) Find the marginal revenue at $x = 2{,}000$, $5{,}000$, and $7{,}000$. Interpret these results.

(E) Graph the cost function and the revenue function in the same coordinate system, find the intersection points of these two graphs, and interpret the results.

(F) Find the profit function and its domain, and sketch its graph.

(G) Find the marginal profit at $x = 1{,}000$, $4{,}000$, and $6{,}000$. Interpret these results.

Solution (A) Since price p and demand x must be nonnegative, we have $x \ge 0$ and

$$p = 10 - 0.001x \ge 0$$
$$10 \ge 0.001x$$
$$10{,}000 \ge x$$

Thus, the permissible values of x are $0 \le x \le 10{,}000$.

(B) The marginal cost is $C'(x) = 2$. Since this is a constant, it costs an additional \$2 to produce one more radio at any production level.

(C) The **revenue** is the amount of money R received by the company for manufacturing and selling x radios at \$$p$ per radio and is given by

$$R = (\text{number of radios sold})(\text{price per radio}) = xp$$

In general, the revenue R can be expressed as a function of p by using equation (1) or as a function of x by using equation (2). As we mentioned earlier, when using marginal functions, we will always use the number of items x as the independent variable. Thus, the **revenue function** is

$$\begin{aligned} R(x) &= xp = x(10 - 0.001x) \quad \text{Using equation (2)} \qquad (4) \\ &= 10x - 0.001x^2 \end{aligned}$$

Since equation (2) is defined only for $0 \le x \le 10{,}000$, it follows that the domain of the revenue function is $0 \le x \le 10{,}000$.

(D) The **marginal revenue** is

$$R'(x) = 10 - 0.002x$$

For production levels of $x = 2{,}000$, $5{,}000$, and $7{,}000$, we have

$$R'(2{,}000) = 6 \qquad R'(5{,}000) = 0 \qquad R'(7{,}000) = -4$$

This means that at production levels of 2,000, 5,000, and 7,000, the respective approximate changes in revenue per unit change in production are \$6, \$0, and −\$4. That is, at the 2,000 output level, revenue increases as production increases; at the 5,000 output level, revenue does not change with a "small" change in production; and at the 7,000 output level, revenue decreases with an increase in production.

(E) When we graph $R(x)$ and $C(x)$ in the same coordinate system, we obtain Figure 3. The intersection points are called the **break-even points** because revenue equals cost at these production levels—the company neither makes nor loses money, but just breaks even. The break-even points are obtained as follows:

$$\begin{aligned} C(x) &= R(x) \\ 7{,}000 + 2x &= 10x - 0.001x^2 \\ 0.001x^2 - 8x + 7{,}000 &= 0 \quad \text{Solve using the quadratic formula} \\ x^2 - 8{,}000x + 7{,}000{,}000 &= 0 \quad \text{(see Appendix A-8).} \end{aligned}$$

$$\begin{aligned} x &= \frac{8{,}000 \pm \sqrt{8{,}000^2 - 4(7{,}000{,}000)}}{2} \\ &= \frac{8{,}000 \pm \sqrt{36{,}000{,}000}}{2} \\ &= \frac{8{,}000 \pm 6{,}000}{2} \\ &= 1{,}000, \quad 7{,}000 \end{aligned}$$

$$\begin{aligned} R(1{,}000) &= 10(1{,}000) - 0.001(1{,}000)^2 = 9{,}000 \\ C(1{,}000) &= 7{,}000 + 2(1{,}000) = 9{,}000 \\ R(7{,}000) &= 10(7{,}000) - 0.001(7{,}000)^2 = 21{,}000 \\ C(7{,}000) &= 7{,}000 + 2(7{,}000) = 21{,}000 \end{aligned}$$

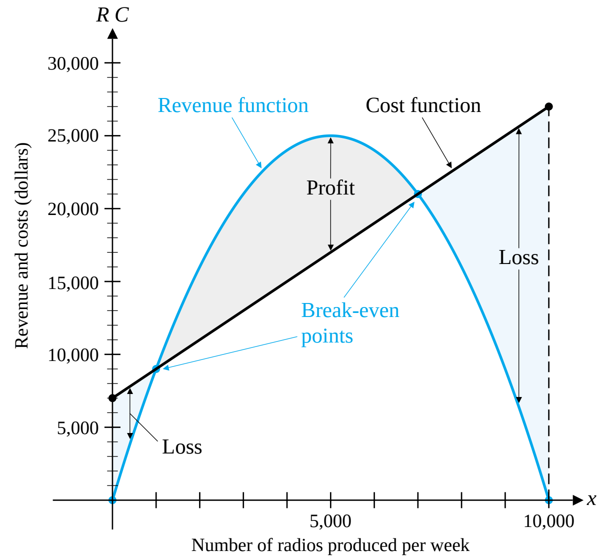

FIGURE 3

Thus, the break-even points are (1,000, 9,000) and (7,000, 21,000), as shown in Figure 3. Further examination of the figure shows that cost is greater than revenue for production levels between 0 and 1,000 and also between 7,000 and 10,000. Consequently, the company incurs a loss at these levels. On the other hand, for production levels between 1,000 and 7,000, revenue is greater than cost and the company makes a profit.

(F) The **profit function** is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (10x - 0.001x^2) - (7{,}000 + 2x) \\ &= -0.001x^2 + 8x - 7{,}000 \end{aligned}$$

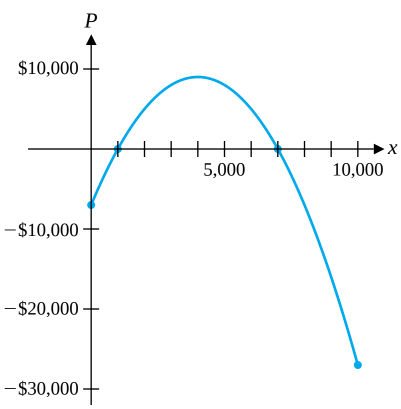

FIGURE 4

The domain of the cost function is $x \geq 0$ and the domain of the revenue function is $0 \leq x \leq 10{,}000$. Thus, the domain of the profit function is the set of x values for which both functions are defined; that is, $0 \leq x \leq 10{,}000$. The graph of the profit function is shown in Figure 4. Notice that the x coordinates of the break-even points in Figure 3 are the x intercepts of the profit function. Furthermore, the intervals where cost is greater than revenue and where revenue is greater than cost correspond, respectively, to the intervals where profit is negative and the intervals where profit is positive.

(G) The **marginal profit** is

$$P'(x) = -0.002x + 8$$

For production levels of 1,000, 4,000, and 6,000, we have

$$P'(1{,}000) = 6 \qquad P'(4{,}000) = 0 \qquad P'(6{,}000) = -4$$

This means that at production levels of 1,000, 4,000, and 6,000, the respective approximate changes in profit per unit change in production are \$6, \$0, and −\$4. That is, at the 1,000 output level, profit will be increased if production is increased; at the 4,000 output level, profit does not change for "small" changes in production; and at the 6,000 output level, profits will decrease if production is increased. It seems the best production level to produce a maximum profit is 4,000. ■

Example 2 warrants careful study, since a number of important ideas in economics and calculus are involved. In the next chapter, we will develop a systematic procedure for finding the production level (and, using the demand equation, the selling price) that will maximize profit.

Matched Problem 2 Refer to the revenue and profit functions in Example 2.

(A) Find $R'(3{,}000)$ and $R'(6{,}000)$, and interpret the results.
(B) Find $P'(2{,}000)$ and $P'(7{,}000)$, and interpret the results.

EXPLORE & DISCUSS 2

Let

$$C(x) = 12{,}000 + 5x \qquad \text{and} \qquad R(x) = 9x - 0.002x^2$$

Explain why $\neq$ is used below. Then find the correct expression for the profit function.

$$P(x) = R(x) - C(x) \neq 9x - 0.002x^2 - 12{,}000 + 5x$$

➤ Marginal Average Cost, Revenue, and Profit

Sometimes, it is desirable to carry out marginal analysis relative to **average cost (cost per unit), average revenue (revenue per unit), and average profit (profit per unit).** The relevant definitions are summarized in the following box:

DEFINITION Marginal Average Cost, Revenue, and Profit

If x is the number of units of a product produced in some time interval, then

Cost per unit: **average cost** $= \overline{C}(x) = \dfrac{C(x)}{x}$

marginal average cost $= \overline{C}'(x) = \dfrac{d}{dx}\overline{C}(x)$

Revenue per unit: **average revenue** $= \overline{R}(x) = \dfrac{R(x)}{x}$

marginal average revenue $= \overline{R}'(x) = \dfrac{d}{dx}\overline{R}(x)$

Profit per unit: **average profit** $= \overline{P}(x) = \dfrac{P(x)}{x}$

marginal average profit $= \overline{P}'(x) = \dfrac{d}{dx}\overline{P}(x)$

EXAMPLE 3 **Cost Analysis** A small machine shop manufactures drill bits used in the petroleum industry. The shop manager estimates that the total daily cost (in dollars) of producing x bits is

$$C(x) = 1{,}000 + 25x - 0.1x^2$$

(A) Find $\overline{C}(x)$ and $\overline{C}'(x)$.
(B) Find $\overline{C}(10)$ and $\overline{C}'(10)$, and interpret.
(C) Use the results in part (B) to estimate the average cost per bit at a production level of 11 bits per day.

Solution (A) $\overline{C}(x) = \frac{C(x)}{x} = \frac{1{,}000 + 25x - 0.1x^2}{x}$

$$= \frac{1{,}000}{x} + 25 - 0.1x \qquad \text{Average cost function}$$

$$\overline{C}'(x) = \frac{d}{dx}\overline{C}(x) = -\frac{1{,}000}{x^2} - 0.1 \qquad \text{Marginal average cost function}$$

(B) $\overline{C}(10) = \frac{1{,}000}{10} + 25 - 0.1(10) = \$124$

$$\overline{C}'(10) = -\frac{1{,}000}{10^2} - 0.1 = -\$10.10$$

At a production level of 10 bits per day, the average cost of producing a bit is \$124, and this cost is decreasing at the rate of \$10.10 per bit.

(C) If production is increased by 1 bit, then the average cost per bit will decrease by approximately \$10.10. Thus, the average cost per bit at a production level of 11 bits per day is approximately $\$124 - \$10.10 = \$113.90$.

Matched Problem 3 Consider the cost function for the production of radios from Example 2:

$$C(x) = 7{,}000 + 2x$$

(A) Find $\overline{C}(x)$ and $\overline{C}'(x)$.

(B) Find $\overline{C}(100)$ and $\overline{C}'(100)$, and interpret.

(C) Use the results in part (B) to estimate the average cost per radio at a production level of 101 radios.

EXPLORE & DISCUSS 3

A student produced the following solution to Matched Problem 3:

$$C(x) = 7{,}000 + 2x \qquad \text{Cost}$$

$$C'(x) = 2 \qquad \text{Marginal cost}$$

$$\frac{C'(x)}{x} = \frac{2}{x} \qquad \text{“Average” of the marginal cost}$$

Explain why the last function is not the same as the marginal average cost function.

Caution

1. The marginal average cost function must be computed by first finding the average cost function and then finding its derivative. As Explore–Discuss 3 illustrates, reversing the order of these two steps produces a different function that does not have any useful economic interpretations.
2. Recall that the marginal cost function has two interpretations: the usual interpretation of any derivative as an instantaneous rate of change, and the special interpretation as an approximation to the exact cost of the $(x + 1)$st item. This special interpretation does not apply to the marginal average cost function. Referring to Example 3, it would be incorrect to interpret $\overline{C}'(10) = -\$10.10$ to mean that the average cost of the next bit is approximately $-\$10.10$. In fact, the phrase “average cost of the next bit” does not even make sense. Averaging is a concept applied to a collection of items, not to a single item.

These remarks also apply to revenue and profit functions.

Answers to Matched Problems

1. (A) $C'(x) = 600 - 1.5x$

(B) $C'(200) = 300$. At a production level of 200 transmissions, total costs are increasing at the rate of \$300 per transmission.

(C) $C(201) - C(200) = \$299.25$.

2. (A) $R'(3{,}000) = 4$. At a production level of 3,000, a unit increase in production will increase revenue by approx. \$4.

$R'(6{,}000) = -2$. At a production level of 6,000, a unit increase in production will decrease revenue by approx. \$2.

(B) $P'(2{,}000) = 4$. At a production level of 2,000, a unit increase in production will increase profit by approx. \$4.

$P'(7{,}000) = -6$. At a production level of 7,000, a unit increase in production will decrease profit by approx. \$6.

3. (A) $\overline{C}(x) = \frac{7{,}000}{x} + 2$; $\overline{C}'(x) = -\frac{7{,}000}{x^2}$

(B) $\overline{C}(100) = \$72$; $\overline{C}'(100) = -\$0.70$. At a production level of 100 radios, the average cost per radio is \$72, and this average cost is decreasing at a rate of \$0.70 per radio.

(C) Approx. \$71.30.