Section 14 Related Rates HW #11

#1 Assume x and y are functions of t. This means that they are expressible as $x(t)$ and $y(t)$. However, the equation given is not explicit in t. Indeed, t does not appear in it.

Given: $xy - x + 2y^3 = -70$

Evaluate: dy/dt, when $dx/dt = -5$, $x=2$, $y=-3$

By implicit differentiation with respect to t; using the product rule on the first term, we get:

$$\frac{dx}{dt}y + x\frac{dy}{dt} - \frac{dx}{dt} + 6y^2\frac{dy}{dt} = 0$$

Substituting the given:

$$-5 \cdot -3 + 2\frac{dy}{dt} - (-5) + 6(-3)^2\frac{dy}{dt} = 0$$

$$\cancel{+15+15} \quad 15 + 5 = -2\frac{dy}{dt} - 54\frac{dy}{dt}$$

$$20 = -56\frac{dy}{dt} \qquad \frac{dy}{dt} = \frac{20}{-56}$$

$$\boxed{\frac{dy}{dt} = \frac{-5}{14}}$$

#2 q thousand watches are sold at p dollars/watch

where $p + q^2 = 144$.

How fast is demand changing (dq/dt) when $q=9$, $p=63$ and price per watch is increasing at a rate of $dp/dt = \$2/\text{wk}$?

Again, p and q are implicit fcns of t.
By implicit differentiation:

$$\frac{dp}{dt} + 2q\,\frac{dq}{dt} = 0$$

Substitute: $2 + 2(9)\frac{dq}{dt} = 0$ $\quad \frac{dq}{dt} = \frac{-2}{18} = -\frac{1}{9}$

The negative indicates a decreasing rate.
The $\frac{1}{9}$ refers to thousands of watches, or

$\frac{1}{9}(1000) \approx 111$ watches/wk decrease in demand

#4. $R(x) = 50x - .4x^2$, $C(x) = 5x + 15$
where x is daily production (and sales).

Given 40 units are produced daily ($x = 40$) and the rate of change of production is 10 units/day ($dx/dt = 10$), find:

a) dR/dt c) dP/dt
b) dC/dt

#4 Con'd

a) R is given as a fcn. of x, which in turn is a ~~fcn~~ fcn. of t. This is the "related rate" of revenue to prod to time.

We need an expression ~~for~~ dR/dt. Clearly we'll need to diff'ate $R(x)$ implicitly w.r.t. time t:

$$R = 50x - .4x^2$$

$$\frac{dR}{dt} = \frac{dR}{dx} \cdot \frac{dx}{dt}$$

$$= (50 - .8x)\frac{dx}{dt}$$

Substituting: $x = 40$, $dx/dt = 10$

$$\frac{dR}{dt} = (50 - .8(40))(10) = \frac{\$180}{\text{day}}$$

b) Cost is also an implicit fcn. of t. It is a fcn. of x which in turn is a fcn. of t.

$$\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt} = 5 \cdot \frac{dx}{dt} = 5(10) = \frac{\$50}{\text{day}}$$

c) Profit = Revenue − Cost $= 50x - .4x^2 - 5x - 15$

$$\frac{dP}{dt} = \frac{dP}{dx} \cdot \frac{dx}{dt}$$ as before, the rate dP/dt is related to dx/dt

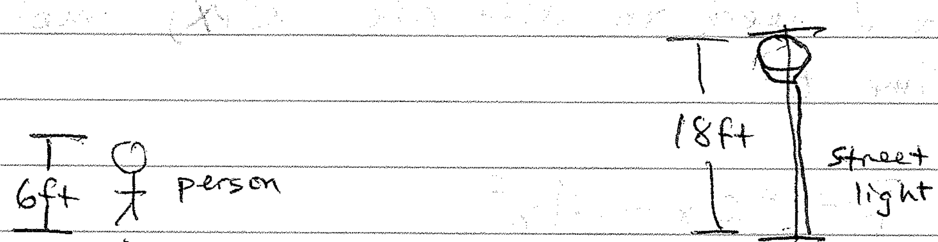
$$P(x) = -.4x^2 + 45x - 15$$

$$\frac{dP}{dt} = \frac{dP}{dx} \cdot \frac{dx}{dt} = -.8x\frac{dx}{dt} + 45\frac{dx}{dt}$$

$$\frac{dP}{dt} = -.8(400) + 45(10) = -320 + 450$$

$$= \$130 / \text{day}$$

#8

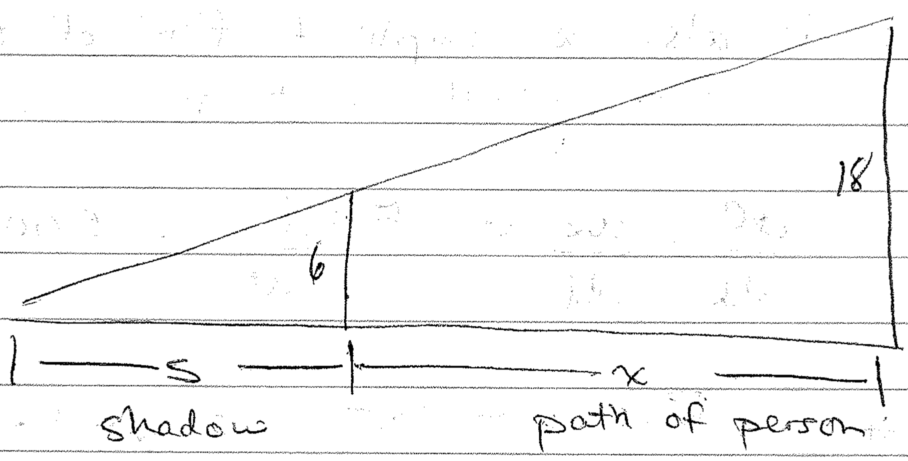


$\leftarrow dx/dt = -5$ ft/sec

(negative indicates direction "away")

The way to interpret "how fast the shadow's tip is moving" is a classic related rate geometry problem.

See Ex. 14.3 for essentially the same problem:



By similar triangles: $\dfrac{x+s}{18} = \dfrac{s}{6}$

We seek $\dfrac{ds}{dt}$ given $\dfrac{dx}{dt}$ and x.

By differentiating $\frac{x+s}{18} = \frac{s}{6}$ implicitly w.r.t. t:

#8 con'd

~~dx/dt~~

$$\frac{dx}{dt} + \frac{ds}{dt} = 3\frac{ds}{dt}$$

$$\frac{dx}{dt} = 2\frac{ds}{dt} \longrightarrow -5 = 2\frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{-5}{2} = -2.5 \frac{ft}{sec} \text{ away}$$

Note: We could have said "away from pole" is positive and, since we'd get the same abs. value but a positive rate for shadow, the relative direction of the shadow would be the same direction for the man.

#9. Number of people $N(p) = p^2 + 5p + 900$
where p has units of 1000 people, $N(p) \equiv$ number seeking treatment

When $p = 20,000$ (call it 20)
and $\frac{dp}{dt} = 1200$ people/year (call it 1.2)

Find $\frac{dN}{dt}$ via $\frac{dN}{dt} = \frac{dN}{dP} \cdot \frac{dP}{dt}$

$$\frac{dN}{dt} = \frac{dN}{dP} \cdot \frac{dP}{dt} = (2p+5)\frac{dP}{dt}$$

$$\frac{dN}{dt} = (2(20)+5)\,\overset{1.2}{\cancel{1200}} = \cancel{45(1200) = 54,000}$$

$= 54$ patients

#12. Product A sales are related to Product B Sales according to $3\sqrt{A} + 5\sqrt{B} = 55$

Find $\frac{dA}{dt}$ when $B = 64$, $\frac{dB}{dt} = 4\ \frac{\text{units}}{\text{day}}$

$$3 \cdot \frac{1}{2} A^{-1/2} \frac{dA}{dt} + 5 \cdot \frac{1}{2} B^{-1/2} \frac{dB}{dt} = 0$$

$$\frac{3}{2} A^{-1/2} \frac{dA}{dt} + \frac{5}{2} (64)^{-1/2} \cdot 4 = 0$$

$$\boxed{\frac{3}{2} A^{-3/2} \frac{dA}{dt} + \frac{20}{2\sqrt{64}} = 0} \longrightarrow \boxed{\frac{3}{2} A^{-3/2} \frac{dA}{dt} + \frac{10}{8} = 0}$$

We seek dA/dt, but we need ~~d~~ A when $B = 64$

From original equation.

$$3\sqrt{A} + 5\sqrt{64} = 0 \longrightarrow \sqrt{A} = -\frac{40}{3}$$

$$\longrightarrow A = \frac{+1600}{9}$$

Substitute this into $\frac{3}{2} A^{-3/2} \frac{dA}{dt} + \frac{10}{8} = 0$

$$\longrightarrow \frac{3}{2} \left(\frac{1600}{9}\right)^{-3/2} \frac{dA}{dt} + \frac{10}{8} = 0$$

$$\longrightarrow \frac{3}{2} \left(\frac{40}{3}\right)^{-3} \frac{dA}{dt} = -\frac{10}{8}$$

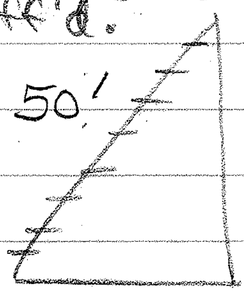
$$\longrightarrow \frac{3}{2} \left(\frac{3}{40}\right)^{3} \frac{dA}{dt} = \frac{-10}{8}$$

Related Rates HW (con'd)

#5. This famous problem uses Pyth. thm. to find both missing side length and generate equation to be diff'd.

(1) Find w

50'



$dw/dt = ?$

$w = ?$

$db/dt = 10'/sec$, $b = 30'$ given

$$50^2 = w^2 + 30^2$$
$$w = \sqrt{2500-900}$$
$$\boxed{w = 40'}$$

(2) Create + solve an eqn.

$$50^2 = w^2 + b^2$$
$$0 = 2w\frac{dw}{dt} + 2b\frac{db}{dt}$$

Substitute what you know:

$$0 = 2(40)\frac{dw}{dt} + 2(30)(10)$$
$$\frac{dw}{dt} = -\frac{60\cancel{0}}{8\cancel{0}}$$
$$\boxed{\frac{dw}{dt} = -7\tfrac{1}{2}\text{ ft/sec}}$$

(downward slide)