Sec. 10 HW #9 Differentiation Rules

1. Prove that for $f(x) = c$, $f'(x) = 0$

Rule 1 $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{c - c}{h}$ ← Do this first

$= \lim\limits_{h \to 0} \dfrac{0}{h} = 0$ Notice h is not added to c because $f(x+h)$ adds h to x terms only

Rule 2 $f(x) = x$, $f'(x) = 1$

Proof: $\lim\limits_{h \to 0} \dfrac{x+h - x}{h} = \lim\limits_{h \to 0} \dfrac{h}{h}$

$= \lim\limits_{h \to 0} 1 = 1$

Rule 4 $(f+g)' = f' + g'$

Right side

<u>Proof</u> $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} + \dfrac{g(x+h) - g(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$

$= \lim\limits_{h \to 0} \dfrac{(f+g)(x+h) - (f+g)(x)}{h}$

$= (f+g)'$ ← Left side

3. Given ~~R~~ $C(x) = 6x^2$, $R(x) = 8x$

Find marginal cost, marginal revenue, marginal profit:

Marginal fcns $C'(x) = 12x$

$R'(x) = 8$

$P'(x) = R'(x) - C'(x) = 12x - 8$.

~~P~~ $P'(x) = 0 = 12x - 8$

Profit = zero when $x = 8/12 = \boxed{2/3}$

4. Demand fcn. ≡ price/unit as a fcn of quantity sold

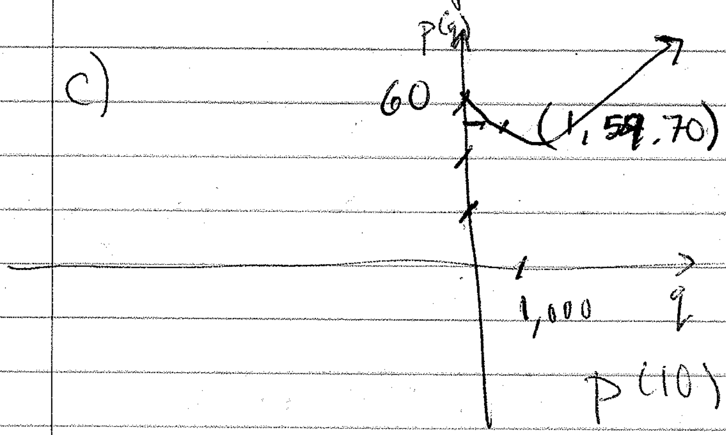
$p(q) = 0.1q^3 - 0.4q + 60$

a) $p'(q) = 0.3q^2 - 0.4 =$

b) $p'(10) = 0.3(10^2) - 0.4 = \$29.60$

That is, \$29.60 is the amount the unit price increases

c)



Unit price p(q)

~~production~~

$p(10) = .1(10)^3 - .4(10) + 60$
$= 96 + 60 = 156$

Sec. 10 cont'd

#4b) Some notes on the solution:
The demand fcn. shows how price (p) charged for an item is a fcn. of demand (q). Initially ($q=0$, nothing sold) the price per unit is \$60. As sales increase, price comes down, but only to a point.

The marginal demand $p'(q)$ shows what the price does for the next unit sold, given a value of q.

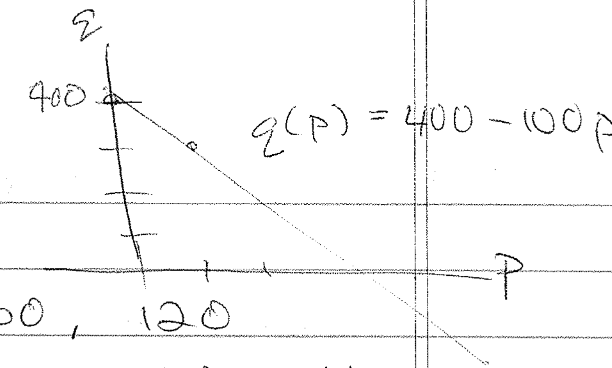
$p'(10) = \$29.60$ indicates that the price will increase \$29.60 for the next 1000 units sold. So, the price will still be increasing as demand increases at this point.

It also asks for the unit price at demand $q = 10$ (10,000 units sold)

Use original function $p(q) = .1q^3 - .4q + 60$

$p(10) = .1(1000) - .4(10) + 60$
$= \$156$ / thousand items sold

5. $q = 400 - 100p$



Find $R'(q)$ for $q = 60,\ 120$ and 400 units

~~Since~~ $R(q) = q \cdot p$ where q is now a constant given (60, for ex); we need p to be in terms of q, so we have to solve $q = 400 - 100p$ for p.

$$100p = 400 - q$$
$$p = 4 - q/100$$

$$R(q) = q \cdot p(q) = q\left(4 - \frac{q}{100}\right) = 4q - \frac{q^2}{100}$$

$$\cancel{R(60) = 240 - \frac{3600}{100} = 204}$$

Now find $R'(q)$, marginal revenue:

$$R'(q) = 4 - \frac{2q}{100} = 4 - q/50$$

$$R'(60) = 4 - 60/50 = 2.8$$
$$R'(120) = 4 - 120/50 = 1.6$$
$$R'(400) = 4 - 400/50 = -4$$

These figures show That revenue for the 61^{st} unit increases by \$2.80, for the 121^{st} by \$1.60 and for 401st it goes down by \$4.

Power rule

#6. a) $f(x) = -1$, $\boxed{f'(x) = 0}$

b) $y = 6x^3 - 3x^2 + 2x + 5$, $\boxed{y' = 18x^2 - 6x + 2}$

c) $f(x) = 2x^{-6}$, $\boxed{f'(x) = -12x^{-7}}$

d) $y = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{3x^3} = x^{-1} - x^{-2} + \frac{x^{-3}}{3}$

$y' = -x^{-2} + 2x^{-3} - \frac{3x^{-4}}{3}$

$\boxed{y' = -x^{-2} + 2x^{-3} - x^{-4}}$

e) $f(r) = \frac{4}{3}\pi r^3$ vol of a sphere

$f'(r) = \frac{4}{3} \cdot 3\pi r^2 = 4\pi r^2$

surface area!

#8. Product rule

a) $(x^2+1)(2x-1) = y$

$\frac{dy}{dx} = (2x)(2x-1) + (x^2+1)(2)$

$= 4x^2 - 2x + 2x^2 + 2$

$= 6x^2 - 2x + 2$

d) $f(x) = xe^x$, $f'(x) = 1 \cdot e^x + x \cdot e^x$

$\boxed{f'(x) = e^x(1+x)}$

#13. given $\begin{cases} f, g \text{ differentiable at } x = 1 \\ f(1) = 1,\ f'(1) = 2,\ g(1) = \frac{1}{2},\ g'(1) = -3 \end{cases}$

a) Find: $(f+g)'(1)$

Since $(f+g)' = f' + g'$, simply add $f'(1) + g'(1)$ to get $2 + -3 = -1$

b) Find $(f-g)'(1)$

Same rule: $(f-g)' = f' - g'$

So $f'(1) - g'(1) = 2 - {}^{-}3 = 5$

c) $(2f + 3g)'(1) = 2f' + 3g' \Big|_{x=1}$

$= 2f'(1) + 3g'(1) = 2(2) + 3(-3) = -5$

d) $(fg)'(1) = f'(1)g(1) + g'(1)f(1)$

$= \cancel{2} \cdot \frac{1}{\cancel{2}} + -3 \cdot 1 = -2$

#15. $p(q) = \dfrac{50}{0.01q^2 + 1} \quad (0 \le q \le 20)$

This is actually an "inverse demand fcn," since q is the independent & p the dependent variable. We could call it a "price fcn."

$\cancel{p'(q) = \dfrac{50(.02q) + 50(.01q^2+1)}{(.01q^2+1)^2}}$

$$p'(q) = \frac{0(.01q^2+1) - 50(.02q)}{(.01q^2+1)^2} = \boxed{\frac{-q}{(.01q^2+1)^2}}$$

a) $p'(5) = \dfrac{-5}{(.01(25)+1)^2} = \dfrac{-5}{(1.25)^2}$

b) $p'(10) = \dfrac{-10}{(1+1)^2} = \dfrac{-10}{4} = -2.5$

$p'(15) = \dfrac{-15}{(.01(15)^2+1)^2} = \cancel{\dfrac{-15}{(1.15)^2}} \quad \dfrac{-15}{(3.25)^2}$

<u>Sec 11</u> Chain rule

2. b) $f(x) = (2x+1)^{-4}$

$f'(x) = -4(2x+1)^{-5}(2) = -8(2x+1)^{-5}$

f) $f(x) = (\sqrt{3}x^2 + x - \sqrt{11})^{-8}$

$f'(x) = -8(\sqrt{3}x^2 + x - \sqrt{11})^{-9}(2\sqrt{3}x + 1)$

j) $f(x) = \left(\dfrac{x^2+x}{1-2x}\right)^4$

$f'(x) = 4\left(\dfrac{x^2+x}{1-2x}\right)^3 \left(\dfrac{(2x+1)(1-2x) - (x^2+x)(-2)}{(1-2x)^2}\right)$

by quotient rule

$= 4\left(\dfrac{x^2+x}{1-2x}\right)^3 \left(\dfrac{-5x^2 + 2x + 1}{(1-2x)^2}\right)$

3a) $f(x) = e^{x+4}$

$f'(x) = e^{x+4}$ since $u = x+4$

$\frac{du}{dx} = 1$

b) $f(x) = 3e^{4x}$

$f'(x) = 3e^{4x} \cdot 4 = 12e^{4x}$

since $u = 4x$

$\frac{du}{dx} = 4$

$$\boxed{\text{Remember; } \frac{d}{dx}(e^u) = e^u \frac{du}{dx}}$$

c) $y = -e^{x+1}$ $\quad y' = -e^{x+1}$ (like a)

d) $f(x) = 30 + 10e^{-0.01x}$, $f'(x) = 10e^{-0.01x} \cdot (-0.01)$

or $\boxed{f'(x) = -.1e^{-.01x}}$

e) $f(x) = 5^{-x}$, $f'(x) = 5^{-x} \cdot (-1) = \boxed{-5^{-x}}$

from $(a^u)' = a^u\, du/dx$

f) $f(x) = xe^x$, $f'(x) = 1e^x + xe^x = \boxed{e^x(1+x)}$

g) $y = (x-3)^2 e^{2x}$

$y' = 2(x-3)e^{2x} + (x-3)^2 e^{2x} \cdot 2$

2r) $f(x) = e^{6x^2+2x+1}$

$f'(x) = \left(e^{6x^2+2x+1}\right)(12x+2)$

#8) Rule 12 says if $f(x) = \log_a x$

then $\boxed{f'(x) = \frac{1}{x \ln a}}$

We did not cover this in class, but using change of base property of logs, the result is clear:

$$f(x) = \log_a x = \frac{\ln x}{\ln a}$$

$$f'(x) = \frac{\ln a \left(\frac{1}{x}\right) - \ln x\,(0)}{(\ln a)^2}$$

$$= \frac{\ln a}{(\ln a)^2 x} = \frac{1}{(\ln a)(x)}$$

I used the quotient rule, but since $\ln a$ is just a coeff, a number, I could have said

$$f'(x) = \frac{1}{\ln a}(\ln x)' = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$$