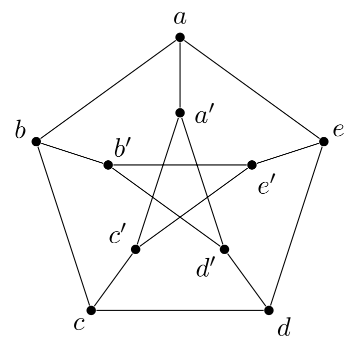
The Petersen Graph is not Hamiltonian

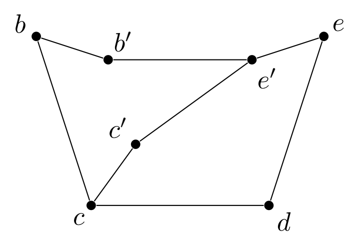
1. Is the Petersen graph a Hamiltonian graph?[1]

Solution: Consider the Petersen graph G with the labelling below:



Assume to the contrary that there is a Hamilton cycle C in G. Observe that by shifting the cycle's starting point we can assume that C starts at a vertex on the outer cycle (i.e. we start at a vertex without a "prime" on it). Now at some point in C we jump from the outer cycle to the inner cycle (i.e. there is a subsequence (x, x') in C). Again shifting (and relabelling if necessary), we can assume C starts with (a, a').

The next vertex in the cycle is either c' or d', so (reflecting the graph if necessary) we may assume that C starts (a, a', d'). Now, the next vertex in the graph is either b' or d. Because C is a Hamilton cycle, in either case both vertices b' and d are the ends of a subpath P of C, so let's consider the possibilities for P; we consider the following graph G', obtained from G by removing the vertices that we have already visited:



Now the paths from d to b' of the desired type are all given below:

$$P_1 = (d, c, b, b') \qquad P_2 = (d, c, c', e', b') \qquad P_3 = (d, e, e', b') \qquad P_4 = (d, e, e', c', c, b, b')$$

Now P is one of the above P_i or their reverses. We finish the proof by cases.

Case 1: If $P = P_1$, then C starts out with either $(a, a', d', d, c, b, b', e')$ or $(a, a', d', b', b, c, d, e)$; the extra vertices at the end are the only available neighbors to move to next. Now in either case we must visit c', which will trap us at c'. Hence this case is impossible.

Case 2: If $P = P_2$, then C starts out with either $(a, a', d', d, c, c', e', b', b)$ or $(a, a', d', b', e', c', c, d, e)$; the extra vertices at the end are the only available neighbors to move to next. In the former case it is impossible to visit e, and in the latter case it is impossible to visit b. Hence this case is impossible.

Case 3: If $P = P_3$, then C starts out with either $(a, a', d', d, e, e', b', b)$ or $(a, a', d', b', e', e, d, c)$; the extra vertices at the end are the only available neighbors to move to next. In either case, we must at some point visit c', which will cut off any remaining route to a. Hence this case is impossible.

Case 4: If $P = P_4$, then C starts out with either $(a, a', d', d, e, e', c', c, b, b')$ or $(a, a', d', b', b, c, c', e', e, d)$. In either case, C has must return to a next; neither b' nor d have an edge to a. Hence this case is impossible.

All cases lead to impossibility, so a Hamilton cycle C cannot possibly exist. Hence G is not Hamiltonian.

[1] It will probably be helpful to draw pictures of the paths we construct below to see what is happening as you go.