Notes on Weak Induction

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Written in predicate logic, the formula for *weak mathematical induction* is:

$$(P(0) \wedge \forall_{k\in\mathbb{N}}[P(k) \to P(k+1)]) \to \forall_{n\in\mathbb{N}}P(n)$$

Given a statement $P(n)$ defined over for all $n \in \mathbb{N}$, to prove $\forall_{n\in\mathbb{N}}P(n)\ldots$

1. Prove $P(0)$ is true. This is the *Base Case*.
2. Prove $P(k) \to P(k+1)$ for all $k \in \mathbb{N}$. This is the *Inductive Step*.

We may then conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

The rest of these notes consists of many examples of the technique above.

Proposition. *For all $n \in \mathbb{N}$ we have*

$$\sum_{j=0}^{n} j = \frac{n(n+1)}{2}.$$

Proof. We proceed by weak mathematical induction on n. For $n \in \mathbb{N}$ let

$$P(n) : \sum_{j=0}^{n} j = \frac{n(n+1)}{2}.$$

Base Case: Notice that $\sum_{j=0}^{0} j = 0 = \frac{0(0+1)}{2}$. Hence $P(0)$ is true and the base case holds.

Inductive Step: Let $k \in \mathbb{N}$ be arbitrary and assume for induction

$$P(k) : \sum_{j=0}^{k} j = \frac{k(k+1)}{2}.$$

By the inductive hypothesis and basic arithmetic we obtain

$$\sum_{j=0}^{k+1} j = \sum_{j=0}^{k} j + (k+1) \overset{IH}{=} \frac{k(k+1)}{2} + (k+1) = \frac{(k)(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

Hence we have shown $P(k+1)$ is true; thus the inductive step holds.

Hence the proposition holds by weak mathematical induction. □

Proposition. *For all $n \in \mathbb{N}$ we have $n < 2^n$.*

Proof. We proceed by weak mathematical induction on n.

Base Case: For $n = 0$ and $n = 1$ we have $0 < 1 = 2^0$ and $1 < 2 = 2^1$. Hence the base case holds.

Inductive Step: Let $k \in \mathbb{N}$ be arbitrary and assume $k < 2^k$; note that we may assume $k \geq 1$. We compute

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Hence the inductive step holds.

Hence the proposition holds by weak mathematical induction. □

Proposition. *For all $n \in \mathbb{N}$ we have*

$$\sum_{j=1}^{n}(2j-1) = n^2.$$

Proof. We proceed by weak mathematical induction on n.

Base Case: We have $\sum_{j=1}^{(0)}(2(0)-1) = 0 = 0^2$, so the base case holds.

Inductive Step: Let $k \in \mathbb{N}$ be arbitrary and assume $\sum_{j=1}^{k}(2j-1) = k^2$. We compute

$$\sum_{j=1}^{k+1}(2j-1) = \sum_{j=1}^{k}(2j-1) + (2(k+1)-1) = (k^2) + 2k + 1 = (k+1)^2.$$

Hence the inductive step holds.

Hence the proposition holds by weak mathematical induction. □

Proposition. *For all $n \in \mathbb{N}$ we have $3 \mid (n^3 - n)$.*

Proof. We proceed by weak mathematical induction on n.

Base Case: For $n = 0$, we have $n^3 - n = 0^3 - 0 = 0 = 3 \cdot 0$. Hence $3 \mid 0^3 - 0$ and the base case holds.

Inductive Step: Let $k \in \mathbb{N}$ be an arbitrary number and assume $3 \mid (k^3 - k)$. By the definition of divisibility, there is an in integer $m \in \mathbb{Z}$ such that $k^3 - k = 3m$. Now we compute

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - k - 1 \\ &= (k^3 - k) + (1 - 1) + 3k^2 + 3k \\ &= 3m + 3k^2 + 3k \\ &= 3(m + k^2 + k)\end{aligned}$$

Now $m + k^2 + k \in \mathbb{Z}$ by closure properties, so $3 \mid ((k+1)^3 - (k+1))$. Hence the inductive step holds.

Hence the proposition holds by weak mathematical induction. □

Proposition. *For all $n \geq 0$, we have $57 \mid (7^{n+2} + 8^{2n+1})$.*

Proof. We proceed by weak mathematical induction on n.

Base Case: Note $57 \cdot 1 = 57 = 49 + 8 = 7^{(0)+2} + 8^{2(0)+1}$, so $57 \mid (7^{(0)+2} + 8^{2(0)+1})$ as desired.

Inductive Step: Let $k \in \mathbb{N}$ be arbitrary number and assume $57 \mid (7^{k+2} + 8^{2k+1})$. By definition of divisibility we have $7^{k+2} + 8^{2k+1} = 57m$ for some $m \in \mathbb{Z}$. Now we compute

$$\begin{aligned}7^{(k+1)+2} + 8^{2(k+1)+1} &= 7^{k+3} + 8^{2k+3} \\ &= 7^{k+2} \cdot 7 + 8^{2k+1} \cdot 8^2 \\ &= ((7^{k+2} \cdot 7) + (8^{2k+1} \cdot 8^2)) \\ &= (7^{k+2} \cdot 7) + ((8^{2k+1}) \cdot (7 + 57)) \\ &= 7(7^{k+2} + 8^{2k+1}) + (8^{2k+1} \cdot 57) \\ &= 7(57m) + (8^{2k+1} \cdot 57) \\ &= 57(7m + 8^{2k+1})\end{aligned}$$

Thus $57 \mid (7^{(k+1)+2} + 8^{2(k+1)+1})$ as $7m + 8^{2k+1} \in \mathbb{Z}$ by closure properties, and the induction step holds.

Hence the original statement holds by weak induction. □