Notes on Binomial Coefficients

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Definition. Let $k, n \in \mathbb{N}$ with $k \leq n$. The *binomial coefficient* $\binom{n}{k} = \#\{S \subseteq [n] : \#S = k\}$.

The notation $\binom{n}{k}$ is read as "n choose k".

Proposition. *For all $n, k \in \mathbb{N}$ with $k \leq n$ we have $2^n = \sum_{k=0}^{n} \binom{n}{k}$.*

Proof. Consider the power set $\operatorname{pow}([n])$. We have $\operatorname{pow}([n]) = \bigsqcup_{k=0}^{n} \{S \subset [n] : \#S = k\}$ which is disjoint union because $S = T$ implies $\#S = \#T$. Therefore by the Sum Principle

$$2^n = \#\operatorname{pow}([n]) = \sum_{k=0}^{n} \#\{S \subset [n] : \#S = k\} = \sum_{k=0}^{n} \binom{n}{k}.$$

Proposition (Pascal's Identity)**.** *For all $n, k \in \mathbb{N}$ with $1 \leq k \leq n$, we have $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.*

Proof. Let $n, k \in N$ with $1 \leq k \leq n$, consider the set $X = \{S \subset [n] : \#S = k\}$. Note that every member $S \in X$ satisfies either $n \in S$ or $n \notin S$. Thus we may express

$$\begin{aligned} X &= \{S \subseteq [n] : \#S = k\} \\ &= \{S \subseteq [n] : \#S = k \text{ and } n \in S\} \sqcup \{S \subseteq [n] : \#S = k \text{ and } n \notin S\} \\ &= \{T \cup \{n\} : \#(T \cup \{n\}) = k \text{ and } T \subseteq [n-1]\} \sqcup \{S \subseteq [n-1] : \#S = k\}. \end{aligned}$$

Now note that the pairing $T \longleftrightarrow T \cup \{n\}$ for each $T \subset [n-1]$ with $\#T = k-1$ is a one-to-one correspondence of the sets $\{T \subseteq [n-1] : \#T = k-1\}$ and $\{T \cup \{n\} : T \subseteq [n-1] \text{ and } \#T = k-1\}$. Hence we apply the Sum Principle and Correspondence Principle to complete the proof by computing

$$\begin{aligned} \#X &= \#\{T \cup \{n\} : T \subset [n-1] \text{ and } \#T = k-1\} + \#\{S \subset [n-1] : \#S = k\} \\ &= \#\{T \subset [n-1] : \#T = k-1\} + \#\{S \subset [n-1] : \#S = k\} \\ &= \binom{n-1}{k-1} + \binom{n-1}{k}. \end{aligned}$$

Proposition. *For all $k, n \in \mathbb{N}$ with $k \leq n$ we have $\binom{n}{k} = \binom{n}{n-k}$.*

Proof. Let $k, n \in \mathbb{N}$ with $k \subseteq [n]$ and define $X = \{S \subset [n] : \#S = k\}$ and $Y = \{S \subset [n] : \#S = n-k\}$. Pair elements of X and Y by the rule $S \longleftrightarrow [n] \setminus S$ for all $S \in X$. Note that $[n] \setminus S = [n] \setminus T$ yields $S = [n] \setminus ([n] \setminus S) = [n] \setminus ([n] \setminus T) = T$, so every element of X is paired to exactly one element of Y and vice versa. Hence the Correspondence Principle yields $\binom{n}{k} = \#X = \#Y = \binom{n}{n-k}$.

Our next order of business is to obtain an algebraic description of the binomial coefficients.

Definition. Let S be a set and $r \in \mathbb{N}$. An *r-permutation* of S is an r-tuple $(s_1, s_2, \ldots, s_r)$ with $s_i \in S$ for all $i \in [r]$ and $s_i = s_j$ implies $i = j$. We also call $(\#S)$-permutations of S just *permutations* of S.

Example 1. Below we write all the r-permutations of $S = [3]$ for $r \in \{0, 1, 2, 3\}$.

r	r-permutations of [3]					
0	$()$					
1	(1)	(2)	(3)			
2	$(1,2)$	$(1,3)$	$(2,1)$	$(2,3)$	$(3,1)$	$(3,2)$
3	$(1,2,3)$	$(1,3,2)$	$(2,1,3)$	$(2,3,1)$	$(3,1,2)$	$(3,2,1)$

Proposition. *Let $r, n \in \mathbb{N}$ with $r \leq n$. Every set of size n has precisely $\frac{n!}{(n-r)!}$ r-permutations.*

To prove this proposition, we will first prove a special case.

Lemma. *Let $n \in \mathbb{N}$. Every n-set has precisely $n!$ permutations.*

Proof of Lemma. We proceed by induction on n.

Base Case: There is a unique 0-permutation, namely $()$. Hence there are $1 = 0!$ permutations of $\emptyset$.

Induction Step: Assume every $(n-1)$-set has exactly $(n-1)!$ permutations. Let S be an arbitrary set of size n. Choose any $s \in S$, and note that $S \setminus \{s\}$ is an $(n-1)$-set. There are $(n-1)!$ permutations of $S \setminus \{s\}$ by the induction hypothesis. Now given a permutation $\sigma = (s_1, s_2, \ldots, s_{n-1})$ of $S \setminus \{s\}$, we build a permutation of S by placing s somewhere in σ. There are $(n-1)!$ options for σ and n positions in which to place s. Hence the Product Principle yields $n(n-1)! = n!$ permutations of S.

We conclude the original statement is true by weak mathematical induction.

Proof of Proposition. Let $r, n \in \mathbb{N}$ with $r \leq n$. Let α denote the number of r-permutations of $[n]$. Note that every n-permutation of $[n]$ is determined by choosing an r-permutation of $[n]$ and then a permutation of the remaining elements. Hence $n! = \alpha(n-r)!$ by the product principle. Solving for α we obtain $\alpha = \frac{n!}{(n-r)!}$.

Proposition. *For all $k, n \in \mathbb{N}$ with $k \leq n$ we have $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$.*

Proof. Let $k, n \in \mathbb{N}$ with $k \leq n$. Every k-permutation of n is obtained by choosing a k-subset $T \subseteq [n]$, and then choosing a permutation of T. Hence $\frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!$, so solving for $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$.

Remark. Arguments of this type are called "counting in two ways" because they prove an equality by enumerating the elements of a single set via two different procedures.

We finish this discussion noting that the name "binomial coefficient" comes from the following theorem.

Proposition (Binomial Theorem)**.** *Let $x, y \in \mathbb{R}$. For all $n \in \mathbb{N}$ we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.*

Proof. Let $x, y \in \mathbb{R}$ be arbitrary. We proceed by induction on n.

Base Case: We have $(x+y)^0 = 1 = \binom{0}{0} x^0 y^0 = \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k}$, verifying the base case.

Induction Step: If $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ for some $n \in \mathbb{N}$, we apply Pascal's Identity to compute

$$\begin{aligned}
(x+y)^{n+1} &= (y+x)(x+y)^n = (x+y)\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k+1} + \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} \\
&= \left[\binom{n}{0}x^0 y^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^k y^{(n+1)-k}\right] + \left[\sum_{k=1}^{n}\binom{n}{k-1}x^k y^{(n+1)-k} + \binom{n}{n}x^{n+1}y^0\right] \\
&= \binom{n+1}{0}x^0 y^{n+1} + \sum_{k=1}^{n}\left[\binom{(n+1)-1}{k} + \binom{(n+1)-1}{k-1}\right] + \binom{n+1}{n+1}x^{n+1}y^0 \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{(n+1)-k}.
\end{aligned}$$

We conclude the original statement is true by weak mathematical induction.

Note that many properties of binomial coefficients follow directly from the binomial theorem. For example

$$2^n = (1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^k 1^{n-k} = \sum_{k=0}^{n}\binom{n}{k},$$

recovering a result we proved earlier. This also allows us to prove purely algebraically many properties of binomial coefficients which are difficult by enumeration-style proofs. For example, for all $n \in \mathbb{Z}^+$ we have

$$0 = ((-1)+1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k},$$

which is rather difficult to prove by enumeration for even $n \in \mathbb{N}$.