Notes on Graph Connection

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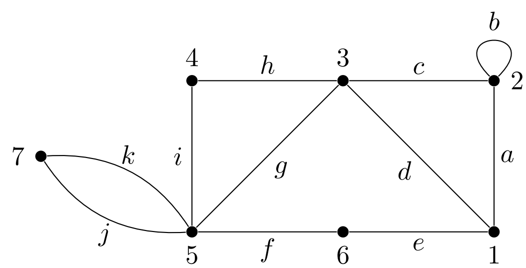
Definition. A *graph* is a structure having vertices and edges, where we allow loops and parallel edges.

Remark. For general graphs, we do allow the following configurations.

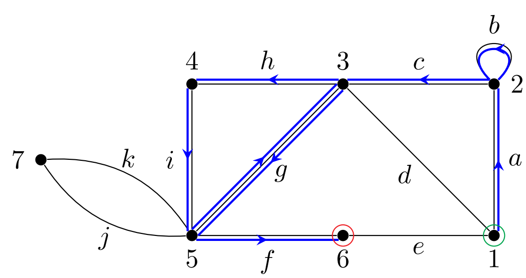
Sometimes we will distinguish between edges with the same endpoints; we do so by labeling the edges.

Example 1. We will use the following graph as a running example in the notes below.



Definition. A *walk* in G is a sequence $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n)$ where $v_i \in V(G)$ for $0 \leq i \leq n$ and e_i is an edge of G with ends v_{i-1} and v_i for all $i \in [n]$.

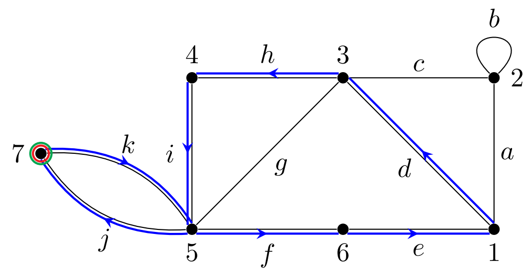
Example 2. The sequence $W = (1, a, 2, b, 2, c, 3, h, 4, i, 5, g, 3, g, 5, f, 6)$ is a walk in the graph of Example 1.



The walk W is highlighted in blue; its start is circled in green and its end is circled in red.

Definition. A *closed walk* in G is a walk which starts and ends at the same vertex. We say a closed walk is *based at* its first vertex.

Example 3. The walk $W = (7, k, 5, f, 6, e, 1, d, 3, h, 4, i, 5, j, 7)$ in the graph of Example 1 is a closed walk.



The walk from Example 2 is not a closed walk as its start and end vertices are distinct.

Definition. The connection relation on graph G is the relation $\sim$ on $V(G)$ defined by $u \sim v$ when there is a walk in G connecting u to v.

Proposition. *Given a graph G, the connection relation is an equivalence relation on $V(G)$.*

Proof. Let G be an arbitrary graph and let $\sim$ denote the connection relation on G.

Reflexive: Let $v \in V(G)$ by arbitrary. Notice that (v) is a walk from v to v in G. Hence $v \sim v$.

Symmetry: Let $u, v \in V(G)$ satisfy $u \sim v$. There is a walk $(u = x_0, e_1, x_1, \ldots, x_n = v)$ in G by definition of $\sim$. Reverse this walk to obtain another walk $(v = x_n, e_{n-1}, x_{n-1}, \ldots, x_1, e_1, x_0 = u)$ in G. As this is a walk in G from v to u. Hence $v \sim u$.

Transitivity: Let $u, v, w \in V(G)$ satisfy $u \sim v$ and $v \sim w$. There are walks $u = (x_0, e_1, x_1, \ldots, x_n = v)$ and $v = (y_0, f_1, y_1, \ldots, y_m = w)$ in G by definition of $\sim$. Concatenating these walks, we obtain a new walk $u = (x_0, e_1, x_1, \ldots, x_n = v = y_0, f_1, y_1, \ldots, y_m = w)$ connecting u to w. Hence $u \sim w$.

Hence $\sim$ is an equivalence relation, as desired. $\square$