Notes on Operations on Relations

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27 March 2020

Recall. A relation from set S to set T is a subset $R \subseteq S \times T$.

Example 1. Consider R from $\{2, 3, 5\}$ to $[20] \setminus [10]$ given by xRy when $x \mid y$, in matrix notation below.

	11	12	13	14	15	16	17	18	19	20
2	0	1	0	1	0	1	0	1	0	1
3	0	1	0	0	1	0	0	1	0	0
5	0	0	0	0	1	0	0	0	0	1

Recall. Let R be a relation on A. Here are some important (possible!) properties of relations.

Name	Property
Reflexivity	$\forall_{x \in A}[xRx]$
Symmetry	$\forall_{x,y \in A}[xRy \to yRx]$
Antisymmetry	$\forall_{x,y \in A}[(xRy \land yRx) \to x = y]$
Transitivity	$\forall_{x,y,z \in A}[(xRy \land yRz) \to xRz]$

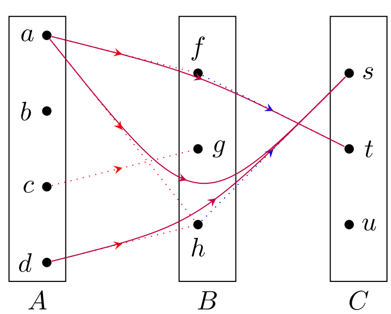
Definition. Let R be a relation from set A to B and Q be a relation from A to C. The *composition* of Q with R is $Q \circ R$ is the relation $Q \circ R \coloneqq \{(a, c) \in A \times C : \text{there is a } b \in B \text{ with } aRb \text{ and } bQc\}$.

The composition relation concatenates arrows in the digraph representations.

Example 2. Let $A = \{a, b, c, d\}$, $B = \{f, g, h\}$, and $C = \{s, t, u\}$. For the relations

$$R = \{(a, f), (a, h), (c, g), (d, h)\} \qquad \text{and} \qquad Q = \{(f, t), (h, s)\},$$

we obtain composition $Q \circ R = \{(a, t), (a, s), (d, s)\}$. The digraph below illustrates the composition; arrows from R are red, arrows from Q are blue, and the arrows of the composition $Q \circ R$ are purple.



Proposition. *Suppose R is a relation on set A. The following are equivalent.*

1. *Relation R is transitive.*
2. *The composition $R \circ R \subseteq R$.*
3. *The n-fold composition $R^n = \underbrace{R \circ R \circ R \circ \cdots \circ R}_{n \text{ times}} \subseteq R$ for all $n \geq 1$.*

A proof of this property is left as an exercise for students to check their understanding of transitivity.

From relation R on set A, we obtain the *transitive closure* $\operatorname{tra}(R) \coloneqq \bigcup_{i=1}^{\infty} R^n$ of R.

Definition. Given any relation R from A to B, the *inverse* relation is $R^{-1} = \{(b,a) \in B \times A : aRb\}$.

The inverse relation reverses the arrows of the digraph representation (what does it do to the matrix?).

Proposition. *A relation R on a set A is symmetric if and only if $R^{-1} \subseteq R$.*

A proof of this property is left as an exercise for students to check their understanding of symmetry.

From relation R on set A, we obtain the *symmetric closure* $\operatorname{sym}(R) \coloneqq R \cup R^{-1}$ of R.

Example 3. Let A be a set. The *identity relation* on A is $\operatorname{id}_A \coloneqq \{(a,a) : a \in A\}$.

Proposition. *A relation R on A is reflexive if and only if $\operatorname{id}_A \subseteq R$.*

Again, this is left for students to check their understanding.

From relation R on set A, we obtain the *reflexive closure* $\operatorname{ref}(R) \coloneqq \operatorname{id}_A \cup R$ of R.

These three "closure operations" allow us to build an equivalence relation from any relation R on A.

Proposition. *Let R be a relation on the set A. The relation $E(R) = \operatorname{ref}(\operatorname{tra}(\operatorname{sym}(R)))$ is an equivalence relation on A. Moreover, if Q is an equivalence relation with $R \subseteq Q$, then $E(R) \subseteq Q$.*

Thus $E(R)$ is the smallest equivalence relation on A containing R.

Proof. Let R be an arbitrary relation on A, and let $E = E(R)$ as above. Note xEy if and only if either $x = y$ or there is an $n \in \mathbb{N}$ and a sequence $(x = a_0, a_1, \cdots, a_n = y)$ such that $(a_{i-1}, a_i) \in R \cup R^{-1}$ for all $i \in [n]$.

Reflexivity: By definition we have $\operatorname{id}_A \subseteq E$, so E is reflexive.

Symmetry: Suppose xEy. If $x = y$, there is nothing to prove. Otherwise there is a sequence

$$(x = w_0, w_1, w_2, \dots, w_n = y)$$

in A such that for all $i \in [n]$ we have $(w_{i-1}, w_i) \in R \cup R^{-1}$. Thus $(y = w_n, w_{n-1}, \dots, w_0 = x)$ is a sequence in A such that for all $i \in [n]$ we have $(w_i, w_{i-1}) \in R \cup R^{-1}$, so yEx. Hence E is symmetric.

Transitivity: Suppose xEy and yEz. If either $x = y$ or $y = z$, there is nothing to prove. Otherwise there are $m, n \in \mathbb{N}$ and sequences $(x = a_0, a_1, \dots, a_m = y)$ and $(y = b_0, b_1, \dots, b_n = z)$ with $(a_{i-1}, a_i) \in R \cup R^{-1}$ for all $i \in [m]$ and $(b_{j-1}, b_j) \in R \cup R^{-1}$ for all $j \in [n]$. Now concatenating these sequences, we obtain a sequence $(x = a_0, a_1, \dots, a_m = y = b_0, b_1, \dots, b_n = z)$; thus xEz. Hence E is transitive.

Hence E is an equivalence relation on A. Now suppose Q is an equivalence relation on A with $R \subseteq Q$. Note $\operatorname{id}_A \subseteq Q$ by reflexivity of Q, and $R^{-1} \subseteq Q^{-1} \subseteq Q$ by symmetry of Q. By transitivity of Q we have

$$\bigcup_{n=1}^{\infty} (R \cup R^{-1}) \subseteq \bigcup_{n=1}^{\infty} Q \subseteq Q.$$

Hence we have $E(R) = \operatorname{ref}(\operatorname{tra}(\operatorname{sym}(R))) = \operatorname{id}_A \cup \bigcup_{n=1}^{\infty} (R \cup R^{-1}) \subseteq Q$ as desired. $\square$