# Notes on Relations

Scribe: Charlie Bartoletti Lecturer/Editor: Chris Eppolito

23 March 2020

**Definition.** A *relation* from set $S$ to set $T$ is a subset $R \subseteq S \times T$.

We often say a relation $R$ from $A$ to $A$ is a *relation on* $A$. Our examples below are written in "pairs notation". It's often cumbersome to write "$(a, b) \in R$"; we often abbreviate this to $aRb$.

**Example 1.** The following are simple examples of relations.

1. Let $S = \{a, b, c, d, e, f\}$ and $T = \{r, s, t, u, v\}$. The set $R = \{(a, v), (a, t), (c, s), (d, s), (d, r), (e, u)\}$ is a relation from $S$ to $T$.
2. Let $C = \{c : c \text{ is a US city}\}$ and $S = \{s : s \text{ is a US state}\}$. Then $R = \{(c, s) : \text{city } c \text{ is in state } s\}$ is a relation from $C$ to $S$.
3. The divisibility relation on $\mathbb{N}$; i.e. $R = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \mid b\}$.
4. Let $P$ denote the set of people. The friends relation on $P$ is $F \coloneqq \{(a, b) \in P \times P : a \text{ is friends with } b\}$.
5. For all sets $S$ and $T$, there is an empty relation $R = \emptyset$ and a complete relation $R = S \times T$ from $S$ to $T$.

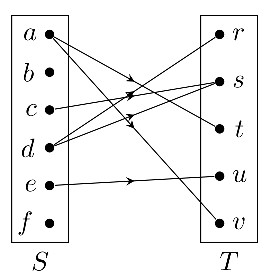
We want more efficient/enlightening representations; two useful notations are "matrix notation" and "digraph notation". The matrix notation associates an array $M$ to the relation; the rows of $M$ are associated to the elements of $S$, and the columns are associated to the elements of $T$. We place a 1 in the $(s, t)$-entry of $M$ when $(s, t) \in R$, and a 0 otherwise. In digraph notation, we line up the elements of $S$ and the elements of $T$ separately (as dots), and draw an arrow from $s \to t$ when $(s, t) \in R$. Both of these notations offer a way to visualize the relation either more compactly (matrix) or with a mind to structure (digraph).

**Example 2.** Let $S = \{a, b, c, d, e, f\}$ and $T = \{r, s, t, u, v\}$, and consider the relation $R$ from $S$ to $T$ below.

$$R = \{(a, v), (a, t), (c, s), (d, s), (d, r), (e, u)\}$$

This relation is expressed in the matrix and digraph notations below.

| | $r$ | $s$ | $t$ | $u$ | $v$ |
|---|---|---|---|---|---|
| $a$ | 0 | 0 | 1 | 0 | 1 |
| $b$ | 0 | 0 | 0 | 0 | 0 |
| $c$ | 0 | 1 | 0 | 0 | 0 |
| $d$ | 1 | 1 | 0 | 0 | 0 |
| $e$ | 0 | 0 | 0 | 1 | 0 |
| $f$ | 0 | 0 | 0 | 0 | 0 |




**Remark.** We have the following.

1. The empty relation is represented by a matrix with all 0's, and a digraph with no arrows.
2. The complete relation is represented by a matrix with all 1's, and a digraph with all possible arrows.

3. If $S = \emptyset = T$, the only relation from $S$ to $T$ is the empty relation $R = \emptyset$. In this case, $R$ is represented by the $0 \times 0$ matrix and the empty digraph.

4. If $S$ and $T$ are finite and $R$ is a relation from $S$ to $T$, then $R$ has a matrix representation; the matrix representation does depend on the order in which you list $S$ and $T$.

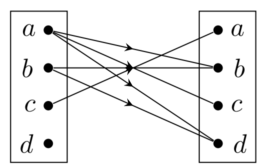
When $R$ is a relation on set $A$, we can represent $R$ as a digraph much more efficiently; rather than write out both sides, we create one dot for each element of $A$ and draw an arrow just among these dots.

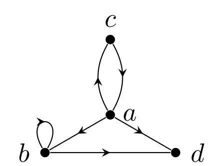
**Example 3.** Let $A = \{a, b, c, d\}$ and consider the relation below.

$$R = \{(a, b), (a, c), (a, d), (b, b), (b, d), (c, a)\}$$

We represent $R$ in all three possible notations below.

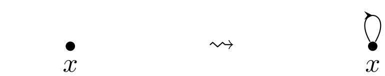
| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0 | 1 | 1 | 1 |
| $b$ | 0 | 1 | 0 | 1 |
| $c$ | 1 | 0 | 0 | 0 |
| $d$ | 0 | 0 | 0 | 0 |



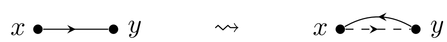




**Definition.** Let $A$ be a set and $R$ a relation on $A$.

1. Relation $R$ is *reflexive* when for all $x \in A$ we have $xRx$.

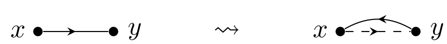



2. Relation $R$ is *symmetric* when for all $x, y \in A$ we have $xRy$ implies $yRx$.




3. Relation $R$ is *transitive* when for all $x, y, z \in A$ we have $xRy$ and $yRz$ implies $xRz$.


4. Relation $R$ is *antisymmetric* when for all $x, y \in A$ we have $xRy$ and $yRx$ implies $x = y$.




**Problem 1.** For the properties above, give a relation satisfying precisely the properties of each subset of the properties (or give a proof that no such relation exists).