Notes on the Pigeonhole Principle

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Definition. A *(weak) partition* of a set S is a collection $S_1, S_2, \cdots, S_n$ of pairwise disjoint subsets of S such that $S = \bigcup_{k=1}^{n} S_k$.

Proposition (Pigeonhole Principle)**.** *Let S be a set and let $\{S_1, S_2, S_3, \ldots, S_k\}$ be a weak partition of S. If $\#S > k$, then there is an index i such that $\#S_i > 1$.*

Remark. This means given $k+1$ objects sorted into k boxes, some box has at least 2 objects.

Example 1. In every collection S of $k \geq 2$ integers, some two are congruent modulo $k-1$.

Solution: There are $k-1$ "boxes" (i.e. *parts*) of the partition $S_i \coloneqq \{a \in S : a \equiv i \pmod{k-1}\}$ for $0 \leq i \leq k-2$, and $\#S = k$. Thus, as there are $k-1$ parts and k integers to consider, there is a i so that $\#S_i > 1$ by the Pigeonhole Principle.

Example 2. A sports team plays games over the course of a 30-day month. They play at least 1 game per day, but no more than 45 total. Show there's some number of consecutive days during which they play exactly 14 games.

Solution: Let a_i denote the number of games played on or before day i. This yields a sequence $a_1, a_2, \ldots, a_{30}$ of positive integers. Note $i \leq a_i < a_{i+1} \leq 45$ for all $i \in [29]$. Consider also the numbers $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$. This yields 60 numbers. Because $1 \leq a_i \leq 45$ for all $i \in [30]$, we obtain $15 \leq a_i + 14 \leq 59$ for all $i \in [30]$. Define

$$b_i \coloneqq \begin{cases} a_i & \text{if } i \in [30] \\ a_{i-30} + 14 & \text{if } i \in [60] \setminus [30]. \end{cases}$$

For all $i \in [60]$ we have $1 \leq b_i \leq 59$. Let $S_k \coloneqq \{i \in [60] : b_i = k\}$ for $1 \leq k \leq 59$. There 59 boxes for 60 elements, so by the Pigeonhole Principle there is an index $k \in [59]$ such that $\#S_k \geq 2$. Note that $a_i \neq a_j$ unless $i = j$ because $a_i < a_{i+1}$ for all $i \in [29]$; thus for distinct $i, j \in S_k$ we may assume $i \leq 30 < j \leq 60$. Thus $b_j = a_{j-30} + 14$ and $b_i = a_i$ has $a_i = k = a_{j-30} + 14$. In particular $a_i - a_{j-30} = 14$, so between days $(j-30)+1$ and i the team played 14 games exactly.

Example 3. Every collection of $n+1$ members of $[2n]$ has some pair related by divisibility.

I.e. given $S = \{a_1, a_2, \ldots, a_{n+1}\} \subseteq [2n]$, there are distinct indices $i, j \in [n+1]$ with $a_i \mid a_j$.

Solution: By the Fundamental Theorem of Arithmetic we may express every member of S as $a_i = 2^{e_i} q_i$ where $e_i \in \mathbb{N}$ and $q_i \in \mathbb{N}$ is odd. Consider the set $T = \{q_1, q_2, \ldots, q_{n+1}\}$. Partition T into $T_k \coloneqq \{i \in [n+1] : q_i = k\}$ for $k \in [2n]$ odd; there are n odd integers in $[2n]$ and $n+1$ indices, so by the Pigeonhole Principle there are distinct indices $i, j \in [n+1]$ with $q_i = k = q_j$. Choosing our notation carefully, we may assume $e_i \leq e_j$ (if they weren't, we would swap the roles of i and j). Hence $a_i = 2^{e_i} k$ and $a_j = 2^{e_j} k = 2^{e_i + n} k = 2^{e_i} k 2^n = a_i 2^n$ for some $n \in \mathbb{N}$, and we obtain $a_i \mid a_j$ as desired.