Limits Examples

Here are solutions to some very simple limit-type questions.

1. Does the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

Solution: Consider the curves $c_k(t) = \langle t, kt\rangle$ where k is an (unspecified) constant; notice $\lim_{t\to 0} c_k(t) = \langle 0, 0\rangle$. If the limit exists, then by a proposition from class $\lim_{t\to 0} f(c_k(t))$ is independent of k. Now

$$\lim_{t\to 0} f(c_k(t)) = \lim_{t\to 0} f(t, kt) = \lim_{t\to 0} \frac{t(kt)}{t^2+(kt)^2} = \lim_{t\to 0} \frac{t^2k}{t^2(1+k^2)} = \lim_{t\to 0} \frac{k}{1+k^2} = \frac{k}{1+k^2},$$

which yields $\lim_{t\to 0} f(c_0(t)) = 0 \neq \frac{1}{2} = \lim_{t\to 0} f(c_1(t))$. Hence the limit does not exist!

2. Does the limit $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{3x-y^2+4}{x+y}$ exist?

Solution: Consider curves $c_k(t) = \langle -1, 1\rangle + \langle t, kt\rangle = \langle t-1, 1+kt\rangle$ for constant k; notice $\lim_{t\to 0} c_k(t) = \langle -1, 1\rangle$. If the limit exists, then by a proposition from class $\lim_{t\to 0} f(c_k(t))$ is independent of k. Now

$$\begin{aligned}\lim_{t\to 0} f(c_k(t)) = \lim_{t\to 0} f(t-1, 1+kt) &= \lim_{t\to 0} \frac{3(t-1)-(1+kt)^2+4}{(t-1)+(1+kt)} \\ &= \lim_{t\to 0} \frac{3t-2kt+k^2t^2}{t+kt} = \lim_{t\to 0} \frac{t(3-2k+k^2t)}{t(1+k)} \\ &= \lim_{t\to 0} \frac{3-2k+k^2t}{1+k} = \frac{3-2k}{1+k},\end{aligned}$$

which yields $\lim_{t\to 0} f(c_0(t)) = 3 \neq -7 = \lim_{t\to 0} f(c_{-2}(t))$. Hence the limit does not exist!