Lagrange Multipliers Example

1. Compute the dimensions of an open-top box with maximum volume and surface area 12.

Solution: Let x, y, and z denote the length, width, and height respectively of such a box. Now we seek to maximize $f(x,y,z) = xyz$ subject to $g(x,y,z) = 0$ where $g(x,y,z) = xy + 2xz + 2yz - 12$. Using the method of Lagrange multipliers, we will attempt to solve the system $\nabla f = \lambda \nabla g$, $g = 0$ where λ is a real number. As $\nabla f = \langle yz, xz, xy \rangle$ and $\nabla g = \langle y + 2z, x + 2z, 2x + 2y \rangle$, this system is as follows:

$$12 = xy + 2xz + 2yz \tag{1}$$
$$yz = \lambda(y + 2z) \tag{2}$$
$$xz = \lambda(x + 2z) \tag{3}$$
$$xy = \lambda(2x + 2y) \tag{4}$$
$$x, y, z \geq 0 \tag{5}$$

Notice that if $x = 0$ or $y = 0$ or $z = 0$, then $f(x,y,z) = 0$; thus we will focus on the case that $x, y, z > 0$.

Multiplying Equations (2), (3), and (4) through by x, y, and z respectively we obtain

$$xyz = \lambda(xy + 2xz) \tag{6}$$
$$xyz = \lambda(xy + 2yz) \tag{7}$$
$$xyz = \lambda(2xz + 2yz). \tag{8}$$

If $\lambda = 0$, then $f(x,y,z) = xyz = 0$; we will assume $\lambda \neq 0$. Comparing (6) and (7) we obtain $xy\lambda + 2xz\lambda = xy\lambda + 2yz\lambda$, which yields $(x - y)z = 0$. Thus $x = y$, because $z \neq 0$. Now Equation (4) reduces to $x^2 = 4x\lambda$, which implies $x(x - 4\lambda) = 0$. Thus $x = 4\lambda = y$ because $x \neq 0$ and $x = y$. Now Equation (2) yields $4\lambda z = \lambda(4\lambda + 2z)$; dividing through by λ and solving for z yields $z = 2\lambda$. Thus we have shown

$$\begin{cases} x = 4\lambda \\ y = 4\lambda \\ z = 2\lambda \end{cases}.$$

Now we must compute λ. Substituting the above expressions into Equation (1), we obtain

$$12 = (4\lambda)(4\lambda) + 2(4\lambda)(2\lambda) + 2(4\lambda)(2\lambda) = 48\lambda^2.$$

Solving this equation we obtain $\lambda = \pm\frac{1}{2}$; we note $\lambda \neq -\frac{1}{2}$ lest $x < 0$. Hence the only potential solution to our system with $f(x,y,z) \neq 0$ has the following form:

$$\lambda = \tfrac{1}{2}, \quad x = 2, \quad y = 2, \quad z = 1$$

Checking against the original system one easily verifies that this is a solution (you should do this).

Thus $f(2,2,1) = 2 \cdot 2 \cdot 1 = 4 > 0$ is the maximum value, and desired box has dimensions $2 \times 2 \times 1$.

2. Let $A > 0$ be fixed. Show that the maximum volume rectangular box with surface area A is a cube.