math314-02-s19 (Eppolito)

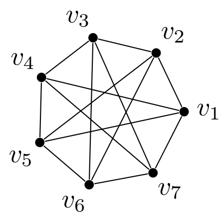
Homework 3

Due 26 April 2019

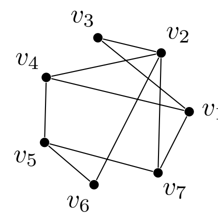
Instructions: Complete each of the following on separate, stapled sheets of paper.

1. Let G be a simple graph with finitely many vertices. Prove that G has an even number of vertices of odd degree.
2. Prove that every graph with at least two vertices has at least two vertices with the same degree.
3. Prove that every subgraph of a bipartite graph is bipartite.
4. For the graphs below, decide whether or not the graph is bipartite. Prove your response correct.

(a)

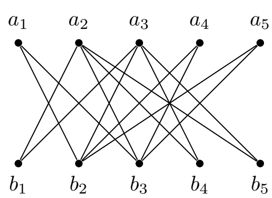


(b)

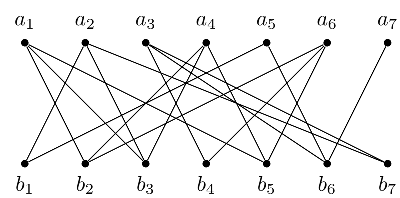


5. Let G be a simple graph with $2n$ vertices and suppose G has at least $\binom{n}{2}^2$ edges. Prove that G is not bipartite.
6. Prove that the complete graph K_{2n} has a perfect matching for all $n \in \mathbb{Z}_{>0}$.
7. For what values of $m, n \in \mathbb{Z}_{>0}$ does the complete bipartite graph $K_{m,n}$ have a perfect matching? Prove it.
8. For each bipartite graph below, use the Augmenting Paths Algorithm to either compute a perfect matching in the graph or find a set of vertices demonstrating the impossibility of this task.

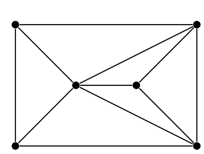
(a)



(b)



9. Let G be a bipartite graph and $r \in \mathbb{Z}_{>0}$. Prove that if G is r-regular, then G has a perfect matching.[1]
10. Let G be a simple graph. Prove that the connection relation in G is an equivalence relation on $V(G)$.
11. Let G be a graph and $v \notin V(G)$. Obtain a new graph $v * G$ from G with vertices $V(v * G) = V(G) \cup \{v\}$ and edges $E(v * G) = E(G) \cup \{\{v, w\} : w \in V(G)\}$. In particular, $v * G$ is the graph obtained by adding a new vertex to G and adding all possible edges from the new vertex to the old vertices.
    (a) Draw $v * C_n$ for $n \in \{3, 4, 5, 6, 7\}$.
    (b) Prove $K_{n+1} = (n + 1) * K_n$ for all $n \in \mathbb{Z}_{+}$.
    (c) Prove $\chi(v * G) = \chi(G) + 1$ for all graphs G.
12. Compute the chromatic number of the graph below.



[1]HINT: Use the Marriage Theorem and the Pigeonhole Principle. Recall that G is r-regular means every vertex of G has degree r.