Functions and Relations Homework

Instructions: Legibly complete each of the following exercises; +1 bonus point if written in LaTeX.

1. Consider the properties "reflexive", "symmetric", "antisymmetric", and "transitive". For each part below either
   i. give a relation R on a set A having precisely these properties and none others from the list, or
   ii. prove no such relation exists.

   You may give your answer in pairs notation, matrix notation, or digraph notation.[1]
   (a) none of the above properties.
   (b) reflexive
   (c) symmetric
   (d) reflexive and symmetric
   (e) antisymmetric
   (f) reflexive and antisymmetric
   (g) symmetric and antisymmetric
   (h) reflexive, symmetric, and antisymmetric
   (i) transitive
   (j) reflexive and transitive
   (k) symmetric and transitive
   (l) reflexive, symmetric, and transitive
   (m) antisymmetric and transitive
   (n) reflexive, antisymmetric, and transitive
   (o) symmetric, antisymmetric, and transitive
   (p) reflexive, symmetric, antisymmetric, and transitive
2. Let R be a relation on a set A.
   (a) Prove that R is reflexive if and only if it contains the relation id_A.
   (b) Prove that R is symmetric if and only if it contains the relation R^{-1}.
   (c) Prove that R is transitive if and only if it contains the relation $\displaystyle\bigcup_{n=1}^{\infty} R^n$.
3. Give examples of functions (as a set) which are...
   (a) both injective and surjective.
   (b) injective but not surjective.
   (c) surjective but not injective.
   (d) neither injective nor surjective.
4. Let $f\colon A \to B$ be a function and let $\equiv$ be an equivalence relation on B. Prove that the relation on A given by $x \sim y$ when $f(x) \equiv f(y)$ is an equivalence relation on A.
5. Let $\sim$ be an equivalence relation on set A; the equivalence class of $a \in A$ is $[a] := \{x \in A : x \sim a\}$. Prove that the set of equivalence classes forms a partition of A (i.e. they form a weak partition of A with no empty part).

[1] Beware of vacuous truth when approaching this problem... I think it's easiest to draw pictures of relations for this problem.