Notes on the Generalized Pigeonhole Principle

Lecturer: Chris Eppolito, Scribe: Kurt Manrique Nino

20 March 2020

Proposition (Generalized Pigeonhole Principle)**.** *Given a weak partition of n objects into k parts, some part has at least $\lceil \frac{n}{k} \rceil$ elements.*

Remark. For $n = k + 1$, we have $\lceil \frac{k+1}{k} \rceil = 2$ elements, recovering the Pigeonhole Principle.

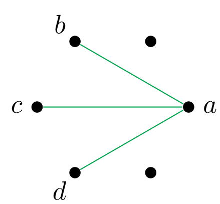
Proposition. *In every group of six people, each pair of which are either mutual friends or mutual enemies, either there are three mutual friends or three mutual enemies among them.*

Below we draw some *graphs* to aid with visualization of the argument; people are represented by black dots, and their relationships are represented by red edges for enemies and green edges for friends. For now these are just visual aids, but later we will study graphs.

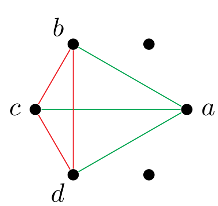
Proof. Suppose we are given six individuals who are pairwise either enemies or friends. Pick any individual a. Individual a is either friends or enemies with each of the other five. Thus a determines a weak partition of the other five individuals into two parts by

$$E = \{x : x \text{ is a friend of } a\} \qquad\qquad F = \{x : x \text{ is an enemy of } a\}.$$

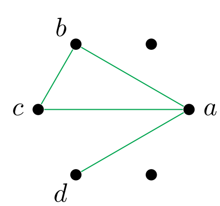
By the Generalized Pigeonhole Principle, there is a part of this weak partition with at least $\lceil \frac{5}{2} \rceil = 3$ members. Thus either a has at least three friends or at least three enemies.



Without loss of generality, we may assume that a has some three friends b, c, and d; the other case follows by switching the roles of friends and enemies. Either b, c, and d are mutual enemies (in which case we're done) or some two of them are friends.

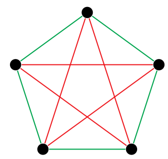


or



If they are not mutual enemies, we may suppose without loss (i.e. by choice of notation) that b and c are friends. Then a, b, and c are three mutual friends. □

Example 1. The following diagram shows a group of five individuals with no three mutual enemies or three mutual friends; thus the above proposition is the best possible.



The above proposition and example amount to a computation of the *Ramsey number* $R(3,3) = 6$. In general $R(m,n)$ is the minimum number of people necessary (with every pair either mutual friends or mutual enemies) to have either m mutual friends or n mutual enemies (for all possible configurations). It is difficult to compute Ramsey numbers in general.

Problem 1 (Optional)**.** Compute the following Ramsey numbers.

(a) $R(2,n)$, where $n \in \mathbb{Z}_{>0}$

(b) $R(3,4)$

Remark. Ramsey numbers are more general than presented here; email me and I can direct you to some resources to learn more!