Instructions: Complete each of the following on separate, stapled sheets of paper.

§12.1: Three-Dimensional Coordinate Systems

1. Find the equation of the sphere centered at the point $(1,2,3)$ and which passes through the origin.
2. Find the side lengths of $\triangle PQR$; classify as right, isosceles, equilateral, or none of these.

   (a) $P=(3,-2,-3),\ Q=(7,0,1),\ R=(1,2,1)$ (b) $P=(2,-1,0),\ Q=(4,1,1),\ R=(4,-5,4)$

3. Determine whether the points below lie on a straight line.

   (a) $A=(2,4,2),\ B=(3,7,-2),\ C=(1,3,3)$ (b) $A=(0,-5,5),\ B=(1,-2,4),\ C=(3,4,2)$

4. Show that the equation below represents a sphere, and find its center and radius.

   (a) $x^2+y^2+z^2-2x-4y+8z=15$ (b) $3x^2+3y^2+3z^2=10+6y+12z$

5. Write inequalities to describe the region below.
   (a) The region between the yz-plane and the vertical plane $x=5$.
   (b) The cylinder on or below plane $z=8$ and on or above the disk about the origin of radius 2 in the xy-plane.
   (c) All points between but not on the spheres of radius r and R centered at the origin, where $r<R$.
   (d) The solid upper hemisphere of the sphere of radius 2 centered at the origin.

§12.2: Vectors

1. Compute the vector given by the directed line segment $\overrightarrow{AB}$ for points A and B given below.

   (a) $A=(0,3,1),\ B=(2,3,-1)$ (b) $A=(0,6,-1),\ B=(3,4,4)$

2. Compute $\vec{u}+\vec{v}$, $2\vec{u}-3\vec{v}$, and $5\vec{v}-\vec{u}$ for $\vec{u},\vec{v}$ below; do so both geometrically and algebraically.

   (a) $\vec{u}=\langle -3,4\rangle,\ \vec{v}=\langle 2,1\rangle$ (b) $\vec{u}=\langle 1,-2\rangle,\ \vec{v}=\langle 1,1\rangle$

3. Find the unit vector in the direction of the given vector.

   (a) $8\vec{i}-\vec{j}+4\vec{k}$ (b) $-5\vec{i}+3\vec{j}-\vec{k}$ (c) $\langle 1,2,3\rangle$

4. Find the vector that has the same direction as $\langle 6,2,-3\rangle$ but has length 4.

§12.3: Dot Product

1. Which of the expressions below are meaningful, and which are meaningless? Explain.
   (a) $(\vec{u} \cdot \vec{v}) \cdot \vec{w}$
   (b) $(\vec{u} \cdot \vec{v})\vec{w}$
   (c) $|\vec{u}|(\vec{v} \cdot \vec{w})$
   (d) $\vec{u} \cdot (\vec{v} + \vec{w})$
   (e) $\vec{u} \cdot \vec{v} + \vec{w}$
   (f) $|\vec{u}| \cdot (\vec{v} + \vec{w})$

2. Compute $\vec{u} \cdot \vec{v}$ for each pair of vectors below.
   (a) $\vec{u} = \langle 5, -2, 3\rangle$, $\vec{v} = \langle 7, 1, 2\rangle$
   (b) $\vec{u} = \langle p, -p, 2p\rangle$, $\vec{v} = \langle 2q, q, -q\rangle$
   (c) $|\vec{u}| = 7$, $|\vec{v}| = 4$, $\theta = 30°$
   (d) $|\vec{u}| = 80$, $|\vec{v}| = 50$, $\theta = \frac{3\pi}{4}$

3. Find the exact angle between the given vectors $\vec{u}$ and $\vec{v}$.
   (a) $\vec{u} = \langle 1, -4, 1\rangle$, $\vec{v} = \langle 0, 2, -2\rangle$
   (b) $\vec{u} = \langle -1, 3, 4\rangle$, $\vec{v} = \langle 5, 2, 1\rangle$
   (c) $\vec{u} = 4\vec{i} - 3\vec{j} + \vec{k}$, $\vec{v} = 2\vec{i} - \vec{k}$
   (d) $\vec{u} = 8\vec{i} - \vec{j} + 4\vec{k}$, $\vec{v} = \vec{j} + 2\vec{k}$

4. Determine whether the given vectors are orthogonal, parallel, or neither.
   (a) $\vec{u} = \langle 4, 5, -2\rangle$, $\vec{v} = \langle 3, -1, 5\rangle$
   (b) $\vec{u} = \langle -5, 4, -2\rangle$, $\vec{v} = \langle 3, 4, -1\rangle$
   (c) $\vec{u} = -8\vec{i} + 12\vec{j} + 4\vec{k}$, $\vec{v} = 6\vec{i} - 9\vec{j} - 3\vec{k}$
   (d) $\vec{u} = 9\vec{i} - 6\vec{j} + 3\vec{k}$, $\vec{v} = -6\vec{i} + 4\vec{j} - 2\vec{k}$

5. Compute the acute angle between the curves below at their points of intersection (the angle between two curves at an intersection point is the angle between the tangent lines at that point).
   (a) $f(x) = x^2$, $g(x) = x^3$
   (b) $f(x) = \sin(x)$, $g(x) = \cos(x)$ for $0 \leq x \leq \frac{\pi}{2}$

6. Find the direction cosines and direction angles of the vector exactly.
   (a) $\langle 2, 1, 2\rangle$
   (b) $\vec{i} - 2\vec{j} - 3\vec{k}$
   (c) $\langle c, c, c\rangle$ for $c > 0$

7. Suppose vector $\vec{v}$ has direction angles $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$. What is the direction angle γ for $\vec{v}$?

8. Find scalar and vector projections of $\vec{v}$ onto $\vec{u}$.
   (a) $\vec{u} = \langle 4, 7, -4\rangle$, $\vec{v} = \langle 3, -1, 1\rangle$
   (b) $\vec{u} = \langle -1, 4, 8\rangle$, $\vec{v} = \langle 12, 1, 2\rangle$
   (c) $\vec{u} = 3\vec{i} - 3\vec{j} + \vec{k}$, $\vec{v} = 2\vec{i} + 4\vec{j} - \vec{k}$
   (d) $\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{v} = 5\vec{i} - \vec{k}$

9. Find the work done by constant force $\vec{F} = \langle 8, -6, 9\rangle$ moving an object from point $P = (0, 10, 8)$ to $Q = (6, 12, 20)$ in a straight line, where force is measured in newtons and distance in meters.

10. A sled is pulled along a level path through snow by a rope. A 30 pound force acts at an angle of 40° above the horizontal and moves the sled 80 feet. Compute the work done by the force.

11. Suppose $\vec{u}$ and $\vec{v}$ are vectors. Prove that if $\vec{u} + \vec{v}$ is orthogonal to $\vec{u} - \vec{v}$, then $\vec{u}$ and $\vec{v}$ have the same length.

12. Let $\vec{u}$ and $\vec{v}$ be vectors. Prove the *Parallelogram Law*: $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2$.

13. Suppose $\vec{u} \neq \vec{0} \neq \vec{v}$ and let $\vec{w} = |\vec{v}|\vec{u} + |\vec{u}|\vec{v}$. Show that $\vec{w} \neq \vec{0}$ implies $\vec{w}$ bisects the angle between $\vec{u}$ and $\vec{v}$.

14. This question walks you through a proof of the Triangle Inequality for vectors. Let $\vec{u}$ and $\vec{v}$ be vectors.
    (a) Prove the *Cauchy-Schwarz Inequality*: $|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|$ (**Hint**: What is $\vec{u} \cdot \vec{v}$ geometrically?).
    (b) Prove the *Triangle Inequality*: $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ (**Hint**: Use the Cauchy-Schwarz Inequality and $|\vec{u} + \vec{v}|^2$).

§12.4: Cross Product

1. Compute the cross product $\vec{u} \times \vec{v}$ and verify it is orthogonal to both $\vec{u}$ and $\vec{v}$.
   (a) $\vec{u} = \langle 4, 3, -2 \rangle$, $\vec{v} = \langle 2, -1, -1 \rangle$
   (b) $\vec{u} = 2\vec{j} - 4\vec{k}$, $\vec{v} = -\vec{i} + 3\vec{j} + \vec{k}$
2. State whether each expression is meaningful or meaningless. Explain.
   (a) $\vec{u} \cdot (\vec{v} \times \vec{w})$
   (b) $\vec{u} \times (\vec{v} \cdot \vec{w})$
   (c) $\vec{u} \times (\vec{v} \times \vec{w})$
   (d) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$
   (e) $(\vec{u} \cdot \vec{v}) \times (\vec{w} \cdot \vec{x})$
   (f) $(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{x})$
3. Find two unit vectors orthogonal to both $\langle 3, 2, 1 \rangle$ and $\langle 1, 2, 3 \rangle$.
4. Find the area of the parallelogram with vertices $A = (-3, 0)$, $B = (-1, 3)$, $C = (5, 2)$, and $D = (3, -1)$.
5. Find a nonzero vector orthogonal to the plane containing $P = (1, 0, 1)$, $Q = (-2, 1, 3)$, and $R = (4, 2, 5)$.
6. Find the volume of the parallelopiped determined by the vectors $\vec{u} = \langle 1, 2, 3 \rangle$, $\vec{v} = \langle -1, 1, 2 \rangle$, and $\vec{w} = \langle 2, 1, 4 \rangle$.
7. Prove the properties of the cross product stated in class.

§12.5: Lines and Planes

1. Determine whether the following statements are true or false in $\mathbb{R}^3$. If false, give an example; if true, explain.
   (a) Two lines parallel to a third line are themselves parallel.
   (b) Two lines perpendicular to a third line are themselves parallel.
   (c) Two planes parallel to a third plane are themselves parallel.
   (d) Two planes perpendicular to a third plane are themselves parallel.
   (e) Two lines parallel to a plane are themselves parallel.
   (f) Two lines perpendicular to a plane are parallel.
   (g) Two planes parallel to a line are themselves parallel.
   (h) Two planes perpendicular to a line are themselves parallel.
   (i) Two planes either intersect or are parallel.
   (j) Two lines either intersect or are parallel.
   (k) A line and a plane either intersect or are parallel.
2. Find vector, parametric, and symmetric equations for the line...
   (a) through points $(1, 1, 2)$ and $(-2, 1, 0)$.
   (b) through $(2, 1, 0)$ and perpendicular to both $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$.
   (c) through $(-6, 2, 3)$ and parallel to line $\frac{1}{2}x = \frac{1}{3}y = z + 1$.
   (d) of intersection for planes $x + 2y + 3z = 1$ and $x - y + z = 1$.
3. Are the lines L_1 and L_2 below parallel, skew, or intersecting? If intersecting, find their point of intersection.
   (a) $L_1 : \langle 3, 4, 1 \rangle + t\langle 2, -1, 3 \rangle$; $L_2 : \langle 1, 3, 4 \rangle + t\langle 4, -2, 5 \rangle$
   (b) $L_1 : x = 5 - 12t, y = 3 + 9t, z = 1 - 3t$; $L_2 : x = 3 + 8t, y = -6t, z = 7 + 2t$
   (c) $L_1 : x - 2 = \dfrac{y - 3}{-2} = \dfrac{z - 1}{-3}$; $L_2 : x - 3 = \dfrac{y + 4}{3} = \dfrac{z - 2}{-7}$
   (d) $L_1 : x = 1 - y = \dfrac{z - 2}{3}$; $L_2 : \dfrac{x - 2}{2} = \dfrac{y - 3}{-2} = \dfrac{z}{7}$

4. Compute an equation of the plane...
   (a) through point $(1, 2, 4)$ and perpendicular to vector $\langle -2, 1, 3\rangle$.
   (b) through point $(2, -5, 1)$ and having normal vector $\langle 1, 4, 1\rangle$.
   (c) through point $(2, 0, 1)$ and perpendicular to line $x = 3t, y = 2 - t, z = 3 + 4t$.
   (d) through point $(1, -1, -1)$ and parallel to plane $5x - y - z = 6$.
   (e) containing line $x = 1 + t, y = 2 - t, z = 4 - 3t$ and parallel to plane $5x + 2y + z = 1$.
   (f) through points $(0, 1, 2)$, $(3, 2, 1)$, and $(1, 1, 2)$.
   (g) through point $(3, 5, -1)$ and containing line $x = 4 - t, y = 2t - 1, z = -3t$.
   (h) through point $(3, 1, 4)$ and containing the line of intersection of planes $x + 2y + 3z = 1$ and $2x - y - z = 2$.
   (i) through points $(0, -2, 5)$ and $(-1, 3, 1)$ and perpendicular to plane $2z = 5x + 4y$.
5. Sketch the plane $2x + 5y + z = 10$.
6. Find the point at which the line $x = 2 - 2t, y = 3t, z = 1 + t$ intersects the plane $x + 2y - z = 7$.
7. Find the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.
8. Are the given planes parallel, perpendicular, or neither?
   (a) $x + 4y - 3z = 1;\ \ -3x + 6y + 7z = 0$
   (b) $9x - 3y + 6z = 2;\ \ 2y = 6x + 4z$
   (c) $x + 2y - z = 2;\ \ 2x - 2y + z = 1$
   (d) $x - y + 3z = 1;\ \ 3x + y - z = 2$
   (e) $2x - 3y = z;\ \ 4x = 3 + 6y + 2z$
   (f) $5x + 2y + 3z = 2;\ \ y = 4x - 6z$
9. Prove the set of points equidistant from points $(x_0, y_0, z_0) \neq (x_1, y_1, z_1)$ forms a plane; compute its equation.

§12.6: Quadratic Surfaces

1. Describe and sketch the surface in $\mathbb{R}^3$ defined by each equation below.
   (a) $4x^2 + y^2 = 4$
   (b) $z = 1 - y^2$
   (c) $xy = 1$
   (d) $z = \sin(y)$
2. Use traces (i.e. cross sections) to sketch and identify the surfaces below.
   (a) $x^2 = 4y^2 + z^2$
   (b) $4x^2 + 9y^2 + 9z^2 = 36$
   (c) $z^2 - 4x^2 - y^2 = 4$
   (d) $9y^2 + 4z^2 = x^2 + 36$
   (e) $3x^2 + y + 3z^2 = 0$
   (f) $y = z^2 - x^2$
3. Reduce $x^2 + y^2 - 2x - 6y - z + 10 = 0$ to standard form, classify the surface, and sketch.

§13.1: Vector Functions and Space Curves

1. Compute the domain of the vector function $\vec{r}(t) = \left\langle \ln(t+1), \dfrac{t}{\sqrt{9-t^2}}, 2^t \right\rangle$.

2. Compute the following limits.

   (a) $\displaystyle\lim_{t\to 0}\left(e^{-3t}\vec{i} + \frac{t^2}{\sin^2(t)}\vec{j} + \cos(2t)\vec{k}\right)$.

   (b) $\displaystyle\lim_{t\to\infty}\left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t}\right\rangle$

3. Sketch the curve with the given vector function. Indicate the direction in which t increases by an arrow.

   (a) $\vec{r}(t) = \langle \sin(t), t\rangle$

   (b) $\vec{r}(t) = \langle \sin(\pi t), t, \cos(\pi t)\rangle$

   (c) $\vec{r}(t) = \langle t, t^2, t^3\rangle$

4. Find vector equation and parametric equations for the line segment joining $(2,-3,1)$ to $(6,2,-2)$.

5. Do Problems 13.1.21 through 13.1.26 in your textbook (matching some parametric equations to pictures).

6. Show that the curve $x = t\cos(t)$, $y = t\sin(t)$, $z = t$ lies on the cone $z^2 = x^2 + y^2$. Make a sketch of the curve.

7. At what points does the helix $\vec{r}(t) = \langle \sin(t), \cos(t), t\rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

8. Find a vector function tracing the curve of intersection of the surfaces given in $\mathbb{R}^3$.

   (a) $S_1 : x^2 + y^2 = 4, \quad S_2 : z = xy$

   (b) $S_1 : z = \sqrt{x^2 + y^2}, \quad S_2 : z = 1 + y$

   (c) $S_1 : z = 4x^2 + y^2, \quad S_2 : y = x^2$

   (d) $S_1 : z = x^2 - y^2, \quad S_2 : x^2 + y^2 = 1$

9. Two particles travel along the curves $\vec{r}_1(t)$ and $\vec{r}_2(t)$ below. Do they collide? Do their paths intersect?

   $$\vec{r}_1(t) = \langle t, t^2, t^3\rangle \qquad \vec{r}_2(t) = \langle 1+2t, 1+6t, 1+14t\rangle.$$

10. Suppose $\vec{u}(t)$ and $\vec{v}(t)$ are vector functions and $f(t)$ is a scalar function. Prove the following.

    (a) $\displaystyle\lim_{t\to a}\left[\vec{u}(t) + \vec{v}(t)\right] = \left[\lim_{t\to a}\vec{u}(t)\right] + \left[\lim_{t\to a}\vec{v}(t)\right]$

    (b) $\displaystyle\lim_{t\to a}\left[f(t)\vec{v}(t)\right] = \left[\lim_{t\to a} f(t)\right]\left[\lim_{t\to a}\vec{v}(t)\right]$

    (c) $\displaystyle\lim_{t\to a}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \left[\lim_{t\to a}\vec{u}(t)\right]\cdot\left[\lim_{t\to a}\vec{v}(t)\right]$

    (d) $\displaystyle\lim_{t\to a}\left[\vec{u}(t)\times\vec{v}(t)\right] = \left[\lim_{t\to a}\vec{u}(t)\right]\times\left[\lim_{t\to a}\vec{v}(t)\right]$

§13.2: Derivatives and Integrals of Vector Functions

1. Sketch the curve $\vec{r}(t)$ in $\mathbb{R}^2$, compute $\vec{r}\,'(t)$, and sketch both $\vec{r}(a)$ and the tangent vector $\vec{r}\,'(a)$.

   (a) $\vec{r}(t) = \langle t^2, t^3\rangle, \ a = 1$

   (b) $\vec{r}(t) = \langle \cos(t)+1, \sin(t)-1\rangle, \ a = -\frac{\pi}{3}$

2. Compute the derivative of the vector function.

   (a) $\vec{r}(t) = \left\langle \sqrt{t-2}, 3, t^{-2}\right\rangle$

   (b) $\vec{r}(t) = t^2\vec{i} + \cos(t^2)\vec{j} + \sin^2(t)\vec{k}$

3. Find the unit tangent vector $\vec{T}(t)$ for $\vec{r}(t)$ at $t = a$.

   (a) $\vec{r}(t) = \left\langle t^2 - 2t, 1+3t, \frac{1}{3}t^3 + \frac{1}{2}t^2\right\rangle, \ a = 2$

   (b) $\vec{r}(t) = \cos(t)\vec{i} + 3t\vec{j} + 2\sin(2t)\vec{k}, \ a = 0$

   (c) $\vec{r}(t) = \langle \arctan(t), 2e^{2t}, 8te^t\rangle, \ a = 0$

   (d) $\vec{r}(t) = \sin^2(t)\vec{i} + \cos^2(t)\vec{j} + \tan^2(t)\vec{k}, \ a = \frac{\pi}{4}$

4. For $\vec{r}(t) = \langle t, t^2, t^3\rangle$ compute $\vec{r}\,'(t)$, $\vec{T}(t)$, $\vec{r}\,''(t)$, $\vec{r}\,'\cdot\vec{r}\,''(t)$, and $\vec{r}\,'\times\vec{r}\,''(t)$.

5. Compute parametric equations of the tangent line to curve $x = t^2+1$, $y = 4\sqrt{t}$, $z = e^{t^2-t}$ at $(2,4,1)$.

6. Find the point on $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), e^2\rangle$ for $0 \le t \le \pi$ with tangent line parallel to plane $\sqrt{3}x + y = 1$.
7. Compute $\displaystyle\int_{t=0}^{1} \left(\frac{1}{t+1}\vec{i} + \frac{1}{t^2+1}\vec{j} + \frac{1}{t^2+1}\vec{k} \right) dt$.
8. Compute $\vec{r}(t)$ supposing $\vec{r}\,'(t) = \langle t, e^t, te^t\rangle$ and $\vec{r}(0) = \vec{i} + \vec{j} + \vec{k}$.
9. Prove that $\frac{d}{dt}\left[\vec{r}(t) \times \vec{r}\,'(t)\right] = \vec{r}(t) \times \vec{r}\,''(t)$ and $|\vec{r}(t)|\frac{d}{dt}\left[\vec{r}(t)\right] = \vec{r}(t) \cdot \vec{r}\,'(t)$ for all vector functions $\vec{r}(t)$.
10. Suppose curve $\vec{r}(t)$ is perpendicular to its tangent $\vec{r}\,'(t)$ for all t. Prove $\vec{r}(t)$ is on a sphere about the origin.

§13.3: Arc Length and Curvature

1. Compute the length of the give curve.
   (a) $\vec{r}(t) = \langle t, 3\cos(t), 3\sin(t)\rangle, \;\; -5 \le t \le 5$
   (b) $\vec{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3\right\rangle, \;\; 0 \le t \le 1$
   (c) $\vec{r}(t) = \sqrt{2}t\,\vec{i} + e^t\,\vec{j} + e^{-t}\,\vec{k}, \;\; 0 \le t \le 1$
   (d) $\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j} + \ln(\cos(t))\,\vec{k}, \;\; 0 \le t \le \frac{\pi}{4}$
   (e) $\vec{r}(t) = \vec{i} + t^2\,\vec{j} + t^3\,\vec{k}, \;\; 0 \le t \le 1$
   (f) $\vec{r}(t) = t^2\,\vec{i} + 9t\,\vec{j} + 4t^{\frac{3}{2}}\,\vec{k}, \;\; 1 \le t \le 4$
2. Parameterize the curve $\vec{r}(t)$ with respect to arc length measured from P in the direction of increasing t.
   (a) $\vec{r}(t) = \langle 5-t, 4t-3, 3t\rangle, \;\; P = (4, 1, 3)$
   (b) $\vec{r}(t) = \langle e^t\sin(t), e^t\cos(t), \sqrt{2}e^t\rangle, \;\; P = (0, 1, \sqrt{2})$
3. Compute unit tangent and unit normal vectors to $\vec{r}(t) = \langle t, 3\cos(t), 3\sin(t)\rangle$; compute the curvature $\kappa(t)$.
4. Compute the curvature $\kappa(t)$ for $\vec{r}(t) = \langle t^2, \ln(t), t\ln(t)\rangle$; what is the curvature at the point $P = (1, 0, 0)$?
5. Compute the unit tangent, unit normal, and binormal vectors of $\vec{r}(t) = \langle \cos(t), \sin(t), \ln(\cos(t))\rangle$.
6. Prove that $\frac{\partial\vec{T}}{\partial s} = \kappa\vec{N}$ for all curves $\vec{r}(t)$.

§13.4: Motion in Space

1. Find the velocity, acceleration, and speed of a particle with the given position function.
   (a) $\vec{r}(t) = \langle t^2+t, t^2-t, t^3\rangle$
   (b) $\vec{r}(t) = \langle \sqrt{2}t, e^t, e^{-t}\rangle$
   (c) $\vec{r}(t) = 3\cos(t)\,\vec{i} + 2\sin(t)\,\vec{j}$
   (d) $\vec{r}(t) = \langle t, 2\cos(t), \sin(t)\rangle$
   (e) $\vec{r}(t) = \langle t, t^2, t^3\rangle$
   (f) $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + 2\,\vec{k}$
2. When is the speed of a particle with position function $\vec{r}(t) = \langle t^2, 5t, t^2-16t\rangle$ minimized?
3. A projectile is fired with initial speed 200 m/s and angle of elevation 60°. Compute the range of the projectile, the maximum height reached, and the speed on impact with the ground.
4. A ball is thrown at an angle of 45° to the ground and lands 90 m away. What is the initial speed of the ball?

§14.1: Functions of Several Variables

1. Consider the functions $f(x, y) = \cos(x + 2y)$ and $g(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z} + \ln(4 - x^2 - y^2 - z^2)$.
   (a) Evaluate $f(2, -1)$ and $g(1, 1, 1)$.
   (b) Compute the domain and range of f.
   (c) Compute the domain and range of g.
2. Sketch graphs of the following functions.
   (a) $f(x, y) = \cos(x)$ (b) $g(x, y) = y$ (c) $h(x, y) = 2 - x^2 - y^2$
3. Complete Problem 14.1.32 in the textbook (matching functions to their graphs).
4. Complete Problem 14.1.38 in the textbook (make a contour map of a given function).
5. Draw a contour map of the functions below, showing several level curves.
   (a) $f(x, y) = x^2 - y^2$ (b) $f(x, y) = xy$ (c) $f(x, y) = \dfrac{y}{x^2 + y^2}$
6. Complete Problems 14.1.61 through 14.1.68 in the textbook (match a function with its graph and contour map).
7. Use a computer to graph $f(x, y) = x^2 + cxy + y^2$ for various $c \in \mathbb{R}$; how does the shape change with changing c?
8. Let $g(t)$ be a single variable function and $f(x, y) = g(\sqrt{x^2 + y^2})$. Describe the graph of f.

§14.2: Limits and Continuity of Multivariate Functions

1. Find the limit (if it exists), or show it does not exist.
   (a) $\displaystyle\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$
   (b) $\displaystyle\lim_{(x,y)\to(2,-1)} \frac{x^2y + xy^2}{x^2 - y^2}$
   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$
   (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2(x)}{x^4 + y^4}$
   (e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$
   (f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + xy + y^2}$
   (g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4 + y^4}$
   (h) $\displaystyle\lim_{(x,y,z)\to(\pi,0,\frac{1}{3})} e^{y^2}\tan(xz)$
   (i) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{x^2 + y^2 + z^2}$
2. Compute $h(x, y) = g(f(x, y))$ and find the set of all points at which h is continuous.
   (a) $g(t) = t^2 + \sqrt{t}, \quad f(x, y) = 2x + 3y - 6$ (b) $g(t) = t + \ln(t), \quad f(x, y) = \dfrac{1 - xy}{1 + x^2y^2}$
3. Determine the set of points at which the function is continuous.
   (a) $f(x, y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$ (b) $g(x, y) = \dfrac{e^x + e^y}{e^{xy} - 1}$ (c) $h(x, y) = \ln(1 + x - y)$
4. Use polar coordinates to compute the limit (**Hint**: As $(x, y) \to (0, 0)$ you can always have $r \to 0^+$...).
   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$
   (b) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2)$
   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$
   (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$
5. Prove that the function $f(\vec{x}) = \vec{v} \cdot \vec{x}$ is continuous on $\mathbb{R}^n$ for all $\vec{v} \in \mathbb{R}^n$.
6. Prove that the function $f(\vec{x}) = |\vec{x}|$ is continuous on $\mathbb{R}^n$. (**Hint**: What type of function is $|\vec{x}|^2$?)

§14.3: Partial Derivatives

1. Use the limit definition of the partial derivative to compute f_x and f_y for the functions $f(x, y)$ below.

   (a) $f(x, y) = xy^2 - x^2y$

   (b) $f(x, y) = \dfrac{x}{x + y^2}$

2. Compute all first order partial derivatives of the given functions.

   (a) $f(x, y) = x^4 + 5xy^3$
   (b) $f(x, y) = x^2y - 3y^4$
   (c) $f(x, t) = t^2e^{-x}$
   (d) $f(x, t) = \sqrt{3x + 4t}$
   (e) $f(x, t) = \ln(x + t^2)$
   (f) $f(x, y) = x\sin(xy)$
   (g) $g(u, v) = (u^2v - v^3)^5$
   (h) $u(r, \theta) = \sin(r\cos(\theta))$
   (i) $f(x, y) = \dfrac{x}{(x + y)^2}$
   (j) $f(x, y) = \dfrac{x}{y}$
   (k) $L(x, y) = \dfrac{ax + by}{cx + dy}$
   (l) $F(x, y) = \displaystyle\int_{t=x}^{y} \cos(e^t)\ dt$
   (m) $u(x_1, x_2, \cdots, x_n) = \sin(x_1 + 2x_2 + \cdots + nx_n)$
   (n) $p(t, u, v) = \sqrt{t^4 + u^2\cos(v)}$
   (o) $f(x, y) = x^y$
   (p) $f(x, y, z) = x^3yz^2 + 2yz$
   (q) $f(x, y, z) = xy^2e^{-xz}$
   (r) $w(x, y, z) = \ln(x + 2y + 3z)$
   (s) $w(x, y, z) = y\tan(x + 2z)$
   (t) $u(x, y, z) = x^{\frac{y}{z}}$
   (u) $R(p, q) = \arctan(pq^2)$
   (v) $w(u, v) = \dfrac{e^v}{u + v^2}$
   (w) $h(x, y, z, t) = x^2y\cos\left(\dfrac{z}{t}\right)$
   (x) $\varphi(x, y, z, t) = \dfrac{\alpha x + \beta y^2}{\gamma z + \delta t^2}$
   (y) $F(\alpha, \beta) = \displaystyle\int_{t=\alpha}^{\beta} \sqrt{t^3 + 1}\ dt$
   (z) $u(x_1, x_2, \cdots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

3. Use implicit differentiation to compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the following implicit surfaces.

   (a) $x^2 + y^2 + z^2 = 1$
   (b) $e^z = xyz$
   (c) $yz + x\ln(y) = z^2$

4. Compute all second order partial derivatives.

   (a) $f(x, y) = x^4y - 2x^3y^2$
   (b) $f(x, y) = \ln(ax + by)$
   (c) $v(s, t) = \sin(s^2 - t^2)$

5. Compute the indicated partial derivatives.

   (a) $f(x, y) = x^4y^2 - x^3y$; $\quad f_{xxx}$, $\quad f_{xyx}$
   (b) $g(r, s, t) = e^r\sin(st)$; $\quad g_{rst}$
   (c) $w(x, y, z) = \dfrac{x}{y + 2z}$; $\quad \dfrac{\partial^3 w}{\partial z\partial y\partial x}$, $\quad \dfrac{\partial^3 w}{\partial x^2\partial y}$

§14.4: Tangent Planes and Linear Approximations

1. Find an equation of the tangent plane to the given surface at the given point.
   (a) $z = 2x^2 + y^2 - 5y$; $P = (1, 2, -4)$
   (b) $z = (x + 2)^2 - 2(y - 1)^2 - 5$; $P = (2, 3, 3)$
   (c) $z = e^{x-y}$; $P = (2, 2, 1)$
   (d) $z = \dfrac{x}{y^2}$; $P = (-4, 2, -1)$
   (e) $z = x \sin(x + y)$; $P = (-1, 1, 0)$
   (f) $z = \ln(x - 2y)$; $P = (3, 1, 0)$

2. Compute the linearization $L(x, y)$ of f at P.
   (a) $f(x, y) = 1 + x \ln(xy - 5)$; $P = (2, 3)$
   (b) $f(x, y) = \sqrt{xy}$; $P = (1, 4)$
   (c) $f(x, y) = x^2 e^y$; $P = (1, 0)$
   (d) $f(x, y) = \dfrac{1 + y}{1 + x}$; $P = (1, 3)$
   (e) $f(x, y) = 4 \arctan(xy)$; $P = (1, 1)$
   (f) $f(x, y) = y + \sin(\frac{x}{y})$; $P = (0, 3)$

3. Compute the differential of the function.
   (a) $z(x, y) = e^{-2x} \cos(2\pi y)$
   (b) $u(x, y) = \sqrt{x^2 + 3y^2}$
   (c) $m(p, q) = p^5 q^3$
   (d) $T(u, v, w) = \dfrac{v}{1 + uvw}$
   (e) $R(\alpha, \beta, \gamma) = \alpha\beta^2 \cos(\gamma)$
   (f) $L(x, y, z) = xze^{-y^2-z^2}$

4. Suppose function $f(x, y)$ is differentiable at (a, b). Prove f is continuous at (a, b).

§14.5: Chain Rule

1. Use the chain rule to compute $\dfrac{df}{dt}$.
   (a) $f(x, y) = xy^3 - x^2 y$; $x(t) = t^2 + 1$, $y(t) = t^2 - 1$
   (b) $f(x, y) = \dfrac{x - y}{x + 2y}$; $x(t) = e^{\pi t}$, $y(t) = e^{-\pi t}$
   (c) $f(x, y) = \sin(x) \cos(y)$; $x(t) = \sqrt{t}$, $y(t) = \dfrac{1}{t}$
   (d) $f(x, y) = \sqrt{1 + xy}$; $x(t) = \tan(t)$, $y(t) = \arctan(t)$
   (e) $f(x, y, z) = x \exp(\dfrac{y}{z})$; $x(t) = t^2$, $y(t) = 1 - t$, $z(t) = 1 + 2t$
   (f) $f(x, y, z) = \ln(\sqrt{x^2 + y^2 + z^2})$; $x(t) = \sin(t)$, $y(t) = \cos(t)$, $z(t) = \tan(t)$

2. Use the chain rule to compute $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.
   (a) $z(x, y) = (x - y)^5$; $x(s, t) = s^2 t$, $y(s, t) = st^2$
   (b) $z(x, y) = \arctan(x^2 + y^2)$; $x(s, t) = s \ln(t)$, $y(s, t) = te^s$
   (c) $z(x, y) = \ln(3x + 2y)$; $x(s, t) = s \sin(t)$, $y(s, t) = t \cos(s)$
   (d) $z(x, y) = \sqrt{x} e^{xy}$; $x(s, t) = 1 + st$, $y(s, t) = s^2 - t^2$
   (e) $z(r, \theta) = e^r \cos(\theta)$; $r(s, t) = st$, $\theta(s, t) = \sqrt{s^2 + t^2}$
   (f) $z(u, v) = \tan(\frac{u}{v})$; $u(s, t) = 2s + 3t$, $v(s, t) = 3s - 2t$

3. A function $f(x, y)$ with $\mathrm{dom}(f) = \mathbb{R}^2$ is *n-homogeneous* when $f(tx, ty) = t^n f(x, y)$ for all t (where $n > 0$).
   (a) Verify that $g(x, y) = x^2 y + 2xy^2 + 5y^3$ is 3-homogeneous.
   (b) Assume f is n-homogeneous and has continuous second-order partial derivatives. Prove the following.
      i. $x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} = nf(x, y)$
      ii. $x^2 \dfrac{\partial^2 f}{\partial x^2} + 2xy \dfrac{\partial^2 f}{\partial x\, \partial y} + y^2 \dfrac{\partial^2 f}{\partial y^2} = n(n - 1)f(x, y)$
      iii. $\dfrac{\partial}{\partial x} [f(tx, ty)] = t^{n-1} f_x(x, y)$

§14.6: Directional Derivatives and the Gradient

1. Compute the directional derivative $D_{\vec{u}} f$ at point P in the direction of angle θ.
   (a) $f(x, y) = xy^3 - x^2$; $P = (1, 2)$, $\theta = \frac{\pi}{3}$
   (b) $f(x, y) = y\cos(xy)$; $P = (0, 1)$, $\theta = \frac{\pi}{4}$
   (c) $f(x, y) = \sqrt{2x + 3y}$; $P = (3, 1)$, $\theta = -\frac{\pi}{6}$
2. Find the directional derivative of f at the point P in the direction of $\vec{v}$.
   (a) $f(x, y) = \dfrac{x}{x^2 + y^2}$; $P = (1, 2)$, $\vec{v} = \langle 3, 5\rangle$
   (b) $f(u, v) = u^2 e^{-v}$; $P = (3, 0)$, $\vec{v} = \langle 3, 4\rangle$
   (c) $f(x, y, z) = x^2 y + y^2 z$; $P = (1, 2, 3)$, $\vec{v} = \langle 2, -1, 2\rangle$
   (d) $f(r, s, t) = \ln(3r + 6s + 9t)$; $P = (1, 1, 1)$, $\vec{v} = \langle 4, 12, 6\rangle$
3. Compute the maximum rate of change of f at point P, and the direction in which it occurs.
   (a) $f(x, y) = 4y\sqrt{x}$; $P = (4, 1)$
   (b) $f(x, y, z) = x\ln(yz)$; $P = (1, 2, \frac{1}{2})$
   (c) $f(x, y, z) = \arctan(xyz)$; $P = (1, 2, 1)$
4. Compute the gradient of the function.
   (a) $f(x, y) = x^2 \ln(y)$ (b) $f(x, y, z) = x^2 yz - xyz^3$ (c) $f(x, y, z) = y^2 e^{xyz}$

§14.7: Minima and Maxima of Two-Variable Functions

1. Compute the local maxima, local minima, and saddle points of the function.
   (a) $f(x, y) = x^2 + xy + y^2 + y$
   (b) $f(x, y) = x^2 + y^4 + 2xy$
   (c) $f(x, y) = y\cos(x)$
   (d) $f(x, y) = xy + e^{-xy}$
   (e) $f(x, y) = xye^{-\frac{x^2+y^2}{2}}$
   (f) $f(x, y) = \sin(x)\sin(y)$
2. Find the absolute maximum and minimum values of the function f on the region R.
   (a) $f(x, y) = x^2 + y^2 - 2x$ on R the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$
   (b) $f(x, y) = x + y - xy$ on R the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$
   (c) $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$ on $R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le 3\}$
   (d) $f(x, y) = xy^2$ on $R = \{(x, y) : 0 \le x,\ 0 \le y,\ x^2 + y^2 \le 3\}$
   (e) $f(x, y) = 2x^3 + y^4$ on $R = \{(x, y) : x^2 + y^2 \le 1\}$
3. Find the points on the cone $z^2 = x^2 + y^2$ closest to $(4, 2, 0)$.
4. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes (i.e. the planes $x = 0$, $y = 0$, and $z = 0$) and with one vertex in the plane $x + 2y + 3z = 6$.
5. Find the maximum volume of a rectangular box which can be inscribed in a sphere of radius r.
6. Find the dimensions of a lidless cardboard box with volume $32,000\ \text{cm}^3$ and minimizing the used cardboard.
7. What is the maximum volume of a rectangular box with diagonal length d?

§14.8: Lagrange Multipliers

1. Use the Lagrange multipliers method to compute the extreme values of f subject to the constraint.
   (a) $f(x, y) = xy; \quad 4x^2 + y^2 = 8$
   (b) $f(x, y) = x^4 + y^4 + z^4; \quad x^2 + y^2 + z^2 = 1$
   (c) $f(x, y, z) = \ln(x^2 + 1) + \ln(y^2 + 1) + \ln(z^2 + 1); \quad x^2 + y^2 + z^2 = 12$
2. Find the extreme values of f subject to the constraints.
   (a) $f(x, y, z) = x + y + z; \quad x^2 + z^2 = 2, \quad x + y = 1$
   (b) $f(x, y, z) = xy + yz; \quad xy = 1, \quad y^2 + z^2 = 1$
   (c) $f(x, y, z) = x^2 + y^2 + z^2; \quad x - y = 1, \quad y^2 - z^2 = 1$
3. Find the points on the curve of intersection of the plane $x + y + 2z = 2$ and the paraboloid $z = x^2 + y^2$ which are nearest the origin and farthest from the origin.

§15.1 – §15.3: Double Integrals

1. Compute the following iterated integrals.
   (a) $\int_{x=1}^{4} \int_{y=0}^{2} (6x^2y - 2x)\, dy\, dx$
   (b) $\int_{y=0}^{1} \int_{x=0}^{1} (x + y)^2\, dx\, dy$
   (c) $\int_{y=0}^{1} \int_{x=1}^{2} (x + e^{-y})\, dx\, dy$
   (d) $\int_{x=0}^{\frac{\pi}{6}} \int_{y=0}^{\frac{\pi}{2}} (\sin(x) + \sin(y))\, dy\, dx$
   (e) $\int_{y=-3}^{3} \int_{x=0}^{\frac{\pi}{2}} \big(y + y^2 \cos(x)\big)\, dx\, dy$
   (f) $\int_{x=1}^{3} \int_{y=1}^{5} \frac{\ln(y)}{xy}\, dy\, dx$
   (g) $\int_{x=1}^{4} \int_{y=1}^{2} \left(\frac{x}{y} + \frac{y}{x}\right) dy\, dx$
   (h) $\int_{y=0}^{1} \int_{x=0}^{2} ye^{x-y}\, dx\, dy$
   (i) $\int_{t=0}^{3} \int_{\phi=0}^{\frac{\pi}{2}} t^2 \sin^3(\sin(\phi))\, d\phi\, dt$
   (j) $\int_{x=0}^{1} \int_{y=0}^{1} xy\sqrt{x^2 + y^2}\, dy\, dx$
   (k) $\int_{v=0}^{1} \int_{u=0}^{1} v(u + v^2)^4\, du\, dv$
   (l) $\int_{t=0}^{1} \int_{s=0}^{1} \sqrt{s + t}\, ds\, dt$
2. Compute the double integral $\iint_R f(x, y)\, dA$ for function $f(x, y)$ and region R.
   (a) $f(x, y) = x \sec^2(y); \quad R = [0, 2] \times [0, \frac{\pi}{4}]$
   (b) $f(x, y) = y + xy^{-2}; \quad R = [0, 2] \times [1, 2]$
   (c) $f(x, y) = \frac{xy^2}{x^2 + 1}; \quad R = [0, 1] \times [-3, 3]$
   (d) $f(x, y) = \frac{\tan(x)}{\sqrt{1 - y^2}}; \quad R = [0, \frac{\pi}{3}] \times [0, \frac{1}{2}]$
   (e) $f(x, y) = x \sin(x + y); \quad R = [0, \frac{\pi}{6}] \times [0, \frac{\pi}{3}]$
   (f) $f(x, y) = \frac{x}{1 + xy}; \quad R = [0, 1] \times [0, 1]$
   (g) $f(x, y) = ye^{-xy}; \quad R = [0, 2] \times [0, 3]$
   (h) $f(x, y) = \frac{1}{1 + x + y}; \quad R = [1, 3] \times [1, 2]$
3. Compute the double integral $\iint_R f(x, y)\, dA$ for function $f(x, y)$ and region R.
   (a) $f(x, y) = \frac{y}{x^2 + 1}; \quad R = \{(x, y) : 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$
   (b) $f(x, y) = 2x + y; \quad R = \{(x, y) : y - 1 \le x \le 1, 1 \le y \le 2\}$
   (c) $f(x, y) = e^{-y^2}; \quad R = \{(x, y) : 0 \le x \le y, 0 \le y \le 3\}$
   (d) $f(x, y) = y\sqrt{x^2 - y^2}; \quad R = \{(x, y) : 0 \le x \le 2, 0 \le y \le x\}$
   (e) $f(x, y) = x \cos(y)$; R the region bounded by $y = 0$, $y = x^2$, and $x = 1$
   (f) $f(x, y) = x^2 + 2y$; R the region bounded by $y = x$, $y = x^3$, and $x \ge 0$
   (g) $f(x, y) = y^2$; R the triangle with vertices $(0, 1)$, $(1, 2)$, and $(4, 1)$
   (h) $f(x, y) = xy$; R the region bounded by $y = \sqrt{1 - x^2}$ with $x, y \ge 0$
   (i) $f(x, y) = 2x - y$; R the radius 2 circular disk about the origin
   (j) $f(x, y) = y$; R the triangle with vertices $(0, 0)$, $(1, 1)$, and $(4, 0)$

4. Sketch the region of integration and then change the order of integration.
   (a) $\int_{y=0}^{1}\int_{x=0}^{y} f(x,y)\,dx\,dy$
   (b) $\int_{x=0}^{\frac{\pi}{2}}\int_{y=0}^{\cos(x)} f(x,y)\,dy\,dx$
   (c) $\int_{x=1}^{2}\int_{y=0}^{\ln(x)} f(x,y)\,dy\,dx$
   (d) $\int_{x=0}^{2}\int_{y=x^2}^{4} f(x,y)\,dy\,dx$
   (e) $\int_{y=-2}^{2}\int_{x=0}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$
   (f) $\int_{x=0}^{1}\int_{y=\arctan(x)}^{\frac{\pi}{4}} f(x,y)\,dy\,dx$

5. Evaluate the integral (**Hint**: Reverse the order of integration).
   (a) $\int_{y=0}^{1}\int_{x=3y}^{3} e^{x^2}\,dx\,dy$
   (b) $\int_{x=0}^{1}\int_{y=x^2}^{1} \sqrt{y}\sin(y)\,dy\,dx$
   (c) $\int_{x=0}^{1}\int_{y=\sqrt{x}}^{1} \sqrt{y^3+1}\,dy\,dx$
   (d) $\int_{y=0}^{2}\int_{x=\frac{1}{2}y}^{1} y\cos(x^3-1)\,dx\,dy$
   (e) $\int_{y=0}^{1}\int_{x=\arcsin(y)}^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2(x)}\,dx\,dy$
   (f) $\int_{y=0}^{8}\int_{x=\sqrt[3]{y}}^{2} e^{x^4}\,dx\,dy$

6. Use a double integral to compute the area of the indicated region.
   (a) One loop of the rose $r(\theta)=\cos(3\theta)$
   (b) The region enclosed by both of the cardioids $r_1(\theta)=1+\cos(\theta)$ and $r_2(\theta)=1-\cos(\theta)$
   (c) The region inside the unit circle centered at $(1,0)$ and outside the unit circle centered at the origin
   (d) The region inside the cardioid $r_1(\theta)=1+\cos(\theta)$ and outside the circle $r_2(\theta)=3\cos(\theta)$

7. Compute the double integral $\iint_R f(x,y)\,dA$ for function $f(x,y)$ and region R.
   (a) $\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$
   (b) $\int_{y=0}^{a}\int_{x=-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\,dx\,dy$
   (c) $\int_{y=0}^{\frac{1}{2}}\int_{x=\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\,dx\,dy$
   (d) $\int_{x=0}^{2}\int_{y=0}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

§15.6 – §15.8: Triple Integrals

1. Evaluate each triple integral $\iiint_R f(x,y,z)\,dV$.
   (a) $\iiint_R y\,dV$ where $R=\{(x,y,z): 0\le x\le 3, 0\le y\le x, x-y\le z\le x+y\}$
   (b) $\iiint_R \exp(zy^{-1})\,dV$ where $R=\{(x,y,z): 0\le y\le 1, y\le x\le 1, 0\le z\le xy\}$
   (c) $\iiint_R \frac{z}{x^2+z^2}\,dV$ where $R=\{(x,y,z): 1\le y\le 4, y\le z\le 4, 0\le x\le z\}$
   (d) $\iiint_R (x-y)\,dV$ where R is bounded by surfaces $z=x^2-1$, $z=1-x^2$, $y=0$, and $y=2$
   (e) $\iiint_R y^2\,dV$ where R is the solid tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, and $(0,0,2)$
   (f) $\iiint_R xz\,dV$ where R is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,1)$, $(0,1,1)$, and $(0,0,1)$
   (g) $\iiint_R x\,dV$ where R is the region bounded by the surfaces $x=4y^2+4z^2$ and $x=4$
   (h) $\iiint_R z\,dV$ where R is the region bounded by $y^2+z^2=9$, $x=0$, $y=3x$, and $z=0$ in the first octant

2. Compute the integral $\iiint_R f(x,y,z)\ dV$ by making a coordinate change to cylindrical coordinates.
   (a) $f(x,y,z) = \sqrt{x^2+y^2}$ and R is the region bounded by $x^2+y^2=16$, $z=-5$, and $z=4$
   (b) $f(x,y,z) = z$ and R is the region bounded by $z = x^2+y^2$ and $z=4$
   (c) $f(x,y,z) = x+y+z$ and R is the region in the first octant bounded by $z = 4-x^2-y^2$
   (d) $f(x,y,z) = x-y$ and R is between cylinders $x^2+y^2=1$ and $x^2+y^2=16$, above $z=0$, and below $z=y+4$
   (e) $f(x,y,z) = x^2$ and R is the region inside $x^2+y^2=1$, above $z=0$, and below $z^2 = 4x^2+4y^2$

3. Compute the integral $\iiint_R f(x,y,z)\ dV$ by making a coordinate change to spherical coordinates.
   (a) $f(x,y,z) = (x^2+y^2+z^2)^2$ and R is the ball of radius 5 about the origin
   (b) $f(x,y,z) = y^2z^2$ and R is the region above the cone $\phi = \frac{\pi}{3}$ and inside the sphere $\rho = 1$
   (c) $f(x,y,z) = x^2+y^2$ and R is the region satisfying $4 \leq x^2+y^2+z^2 \leq 9$
   (d) $f(x,y,z) = y^2$ and R is the solid hemisphere of $x^2+y^2+z^2 \leq 9$ with $y \geq 0$
   (e) $f(x,y,z) = x\exp(x^2+y^2+z^2)$ and R is the portion of the unit ball $x^2+y^2+z^2 \leq 1$ in the first octant
   (f) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ and R is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ and above the cone $z = \sqrt{x^2+y^2}$

4. Compute the volume of the solid R...
   (a) within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$
   (b) in both the sphere $x^2+y^2+z^2=2$ and the cone $z=\sqrt{x^2+y^2}$
   (c) between the paraboloid $z=x^2+y^2$ and the sphere $x^2+y^2+z^2=2$
   (d) between the paraboloid $z = 24-x^2-y^2$ and the cone $z=\sqrt{x^2+y^2}$
   (e) cut out by the cylinder $r = a\cos(\theta)$ and sphere of radius $a>0$ centered at the origin
   (f) lying above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 4\cos(\phi)$
   (g) within the sphere $x^2+y^2+z^2=4$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$
   (h) above the cone $z=\sqrt{x^2+y^2}$ and within the sphere $x^2+y^2+z^2=1$

5. Compute the center of mass of the solid bounded by the paraboloid $z=4x^2+4y^2$ and the plane $z=a$ for $a>0$, supposing the solid has constant density K.

6. Compute the mass of the ball B of radius r about the origin if the density at a point is proportional to its distance from the z-axis.

7. Compute the average distance from a point in a ball of radius r to its center.

§15.9: Changes of Coordinate

1. Compute the (signed) Jacobian of the transformation.

(a) $\begin{cases} x = 2u+v \\ y = 4u-v \end{cases}$

(b) $\begin{cases} x = u^2+uv \\ y = uv^2 \end{cases}$

(c) $\begin{cases} x = uv \\ y = vw \\ z = wu \end{cases}$

(d) $\begin{cases} x = s\cos(t) \\ y = t\cos(s) \end{cases}$

(e) $\begin{cases} x = pe^q \\ y = qe^p \end{cases}$

(f) $\begin{cases} x = u+vw \\ y = v+wu \\ z = w+uv \end{cases}$

2. Use the given transformation to evaluate the integral $\iint_R f(x,y)\,dA$.

   (a) $f(x,y) = x - 3y$;
   R is the triangular region with vertices $(0,0)$, $(2,1)$, and $(1,2)$;
   $\begin{cases} x = 2u + v \\ y = u + 2v \end{cases}$

   (b) $f(x,y) = 4x + 8y$;
   R is the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$;
   $\begin{cases} x = \frac{1}{4}(u+v) \\ y = \frac{1}{4}(v-3u) \end{cases}$

   (c) $f(x,y) = x^2$;
   R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$;
   $\begin{cases} x = 2u \\ y = 3v \end{cases}$

   (d) $f(x,y) = x^2 - xy + y^2$;
   R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$;
   $\begin{cases} x = \sqrt{2}u - \sqrt{\frac{2}{3}}v \\ y = \sqrt{2}u + \sqrt{\frac{2}{3}}v \end{cases}$

   (e) $f(x,y) = xy$;
   R is the region in the first quadrant bounded by $y = x$, $y = 3x$, $xy = 1$, and $xy = 3$;
   $\begin{cases} x = \dfrac{u}{v} \\ y = v \end{cases}$

   (f) $f(x,y) = y^2$;
   R is the region bounded by $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$;
   $\begin{cases} u = xy \\ v = xy^2 \end{cases}$

3. Compute $\iint_R f(x,y)\,dA$ by making an appropriate change of variables.

   (a) $f(x,y) = \dfrac{x-2y}{3x-y}$;
   R is the parallelogram given by $0 \le x - 2y \le 2$ and $1 \le 3x - y \le 8$

   (b) $f(x,y) = (x+y)\exp(x^2 - y^2)$;
   R is the rectangle given by $0 \le x - y \le 2$ and $0 \le x + y \le 3$

   (c) $f(x,y) = \cos\left(\dfrac{y-x}{x+y}\right)$;
   R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$

   (d) $f(x,y) = \sin(9x^2 + 4y^2)$;
   R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$

   (e) $f(x,y) = \exp(x+y)$;
   R is the region satisfying inequality $|x| + |y| \le 1$

4. Prove $\iint_R f(x+y)\,dA = \int_{uf(u)=u}^{0} 1\,du f(u)$ for every continuous function f on $[0,1]$ and R the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

§16.2 - §16.3: Line Integrals

1. Compute the line integral $\int_C f\ ds$.

   (a) $f(x,y) = y$;
   C: $\langle t^2, 2t\rangle$ for $0 \le t \le 3$

   (b) $f(x,y) = \frac{x}{y}$;
   C: $\langle t^3, t^4\rangle$ for $1 \le t \le 2$

   (c) $f(x,y) = xy^4$;
   C: the right half of the circle $x^2 + y^2 = 16$

   (d) $f(x,y) = xe^y$;
   C: the line segment from $(2,0)$ to $(5,4)$

   (e) $f(x,y,z) = x^2y$;
   C: $\langle \cos(t), \sin(t), t\rangle$ for $0 \le t \le \frac{\pi}{2}$

   (f) $f(x,y,z) = x^2 + y^2 + z^2$;
   C: $\langle t, \cos(2t), \sin(2t)\rangle$ for $0 \le t \le 2\pi$

   (g) $f(x,y,z) = y^2z$;
   C: the line segment from $(3,1,2)$ to $(1,2,5)$

   (h) $f(x,y,z) = x\exp(yz)$;
   C: the line segment from $(0,0,0)$ to $(1,2,3)$

2. Compute the indicated line integral directly.

   (a) $\int_C (x^2y + \sin(x))\ dy$;
   C: the arc of $y = x^2$ from $(0,0)$ to (π, π^2)

   (b) $\int_C e^x\ dx$;
   C: the arc of $y = x^3$ from $(-1,-1)$ to $(1,1)$

   (c) $\int_C xye^{yz}\ dy$;
   C: $\vec{r}(t) = \langle t, t^2, t^3\rangle$ for $0 \le t \le 1$

   (d) $\int_C y\ dx + z\ dy + x\ dz$;
   C: $\vec{r}(t) = \langle \sqrt{t}, t, t^2\rangle$ for $1 \le t \le 4$

   (e) $\int_C z^2\ dx + x^2\ dy + y^2\ dz$;
   C: the segment from $(1,0,0)$ to $(4,1,2)$

3. Determine whether or not $\vec{F}$ is a conservative vector field. If it is, compute a potential function for $\vec{F}$.

   (a) $\vec{F} = \langle xy + y^2, x^2 + 2xy\rangle$

   (b) $\vec{F} = \langle y^2 - 2x, 2xy\rangle$

   (c) $\vec{F} = \langle y^2e^{xy}, (1+xy)e^{xy}\rangle$

   (d) $\vec{F} = \langle ye^x, e^x + e^y\rangle$

   (e) $\vec{F} = \langle ye^x + \sin(y), e^x + x\cos(y)\rangle$

   (f) $\vec{F} = \langle y^2\cos(x) + \cos(y), 2y\sin(x) - x\sin(y)\rangle$

   (g) $\vec{F} = \left\langle \ln(y) + \frac{y}{x}, \ln(x) + \frac{x}{y}\right\rangle$

4. Use the Fundamental Theorem of Line Integrals to evaluate $\int_C \vec{F} \cdot d\vec{r}$.

   (a) $\vec{F} = \langle 3 + 2xy^2, 2x^2y\rangle$
   C: the arc of $y = \frac{1}{x}$ from $(1,1)$ to $(4, \frac{1}{4})$

   (b) $\vec{F} = \langle x^2y^3, x^3y^2\rangle$
   C: $\vec{r}(t) = \langle t^3 - 2t, t^3 + 2t\rangle$ for $0 \le t \le 1$

   (c) $\vec{F} = \langle (1+xy)e^{xy}, x^2e^{xy}\rangle$
   C: $\vec{r}(t) = \langle \cos(t), 2\sin(t)\rangle$ for $0 \le t \le \frac{\pi}{2}$

   (d) $\vec{F} = \langle xy, xz, xy + 2z\rangle$
   C: the segment from $(1,0,-2)$ to $(4,6,3)$

   (e) $\vec{F} = \langle y^2z + 2xz^2, 2xyz, xy^2 + 2x^2z\rangle$
   C: $\vec{r}(t) = \left\langle \sqrt{t}, t+1, t^2\right\rangle$ for $0 \le t \le 1$

   (f) $\vec{F} = \langle yze^{xz}, e^xz, xye^{xz}\rangle$
   C: $\vec{r}(t) = \langle t^2 + 1, t^2 - 1, t^2 - 2t\rangle$ for $0 \le t \le 2$

   (g) $\vec{F} = \langle \sin(y), x\cos(y) + \cos(z), -y\sin(z)\rangle$
   C: $\vec{r}(t) = \langle \sin(t), t, 2t\rangle$ for $0 \le t \le \frac{\pi}{2}$

§16.4: Green's Theorem

1. Evaluate the line integral $\int_{\partial R} P\ dx + Q\ dy$ via Green's Theorem.

   (a) $P(x,y) = ye^x,\ \ Q(x,y) = 2e^x$;
   $R = [0,3] \times [0,4]$

   (b) $P(x,y) = x^2 + y^2,\ \ Q(x,y) = x^2 - y^2$;
   R: triangle with vertices $(0,1)$, $(2,1)$, and $(0,1)$

   (c) $P(x,y) = y + e^{\sqrt{x}},\ \ Q(x,y) = 2x + \cos(y^2)$;
   R: the region bounded by $y = x^2$ and $x = y^2$

   (d) $P(x,y) = y^4,\ \ Q(x,y) = 2xy^3$;
   R: the ellipse $x^2 + 2y^2 \leq 2$

   (e) $P(x,y) = y^3,\ \ Q(x,y) = -x^3$;
   R: the ball $x^2 + y^2 \leq 4$

   (f) $P(x,y) = 1 - y^3,\ \ Q(x,y) = x^3 + \exp(y^2)$;
   R: the annulus $4 \leq x^2 + y^2 \leq 9$

2. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ via Green's Theorem.

   (a) $\vec{F} = \langle y\cos(x) - xy\sin(x), xy + x\cos(x)\rangle$
   C: the triangle with vertices $(0,0)$, $(0,4)$, $(2,0)$

   (b) $\vec{F} = \left\langle e^{-x} + y^2, e^{-y} + x^2\right\rangle$
   C: the arc of $y = \cos(x)$ from $(-\frac{\pi}{2}, 0)$ to $(\frac{\pi}{2}, 0)$ and then the segment connecting $(\frac{\pi}{2}, 0)$ to $(-\frac{\pi}{2}, 0)$

   (c) $\vec{F} = \langle y - \cos(y), x\sin(y)\rangle$
   C: the clockwise circle $(x-3)^2 + (y+4)^2 = 4$

   (d) $\vec{F} = \left\langle \sqrt{x^2+1}, \arctan(x)\right\rangle$
   C: the triangle with vertices $(0,0)$, $(1,1)$, $(0,1)$

§16.5: Curl and Divergence

1. Compute the curl and divergence of the vector field $\vec{F}$.

   (a) $\vec{F} = \left\langle xy^2z^2, x^2yz^2, x^2y^2z\right\rangle$

   (b) $\vec{F} = \left\langle 0, x^3yz^2, y^4z^3\right\rangle$

   (c) $\vec{F} = \langle \sin(yz), \sin(zx), \sin(xy)\rangle$

   (d) $\vec{F} = \langle \ln(2y+3z), \ln(x+3z), \ln(x+2y)\rangle$

   (e) $\vec{F} = \langle e^x\sin(y), e^y\sin(z), e^z\sin(x)\rangle$

   (f) $\vec{F} = \langle \arctan(xy), \arctan(yz), \arctan(zx)\rangle$

2. Determine whether or not $\vec{F}$ is conservative. If so, compute a potential f.

   (a) $\vec{F} = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2\right\rangle$

   (b) $\vec{F} = \left\langle xyz^4, x^2z^4, 4x^2yz^3\right\rangle$

   (c) $\vec{F} = \langle z\cos(y), xz\sin(y), x\cos(y)\rangle$

   (d) $\vec{F} = \langle 1, \sin(z), y\cos(z)\rangle$

   (e) $\vec{F} = \langle e^{yz}, xze^{yz}, xye^{yz}\rangle$

   (f) $\vec{F} = \langle e^x\sin(yz), ze^x\cos(yz), ye^x\cos(yz)\rangle$

3. Is there a vector field $\vec{G}$ on $\mathbb{R}^3$ for which $\text{curl}(\vec{G}) = \vec{F}$? Why?

   (a) $\vec{F} = \langle x\sin(y), \cos(y), z - xy\rangle$

   (b) $\vec{F} = \langle x, y, z\rangle$

4. Prove every vector field of form $\vec{F}(x,y,z) = \langle f(x), g(y), h(z)\rangle$ is *irrotational* (i.e. $\text{curl}(\vec{F}) = \vec{0}$).

5. Prove every vector field of form $\vec{F}(x,y,z) = \langle f(y,z), g(x,z), h(x,y)\rangle$ is *incompressible* (i.e. $\text{div}(\vec{F}) = 0$).

6. Prove each of the following identities for scalar field $\alpha(x,y,z)$ vector fields $\vec{F}(x,y,z)$ and $\vec{G}(x,y,z)$.

   (a) $\text{div}(\vec{F} + \vec{G}) = \text{div}(\vec{F}) + \text{div}(\vec{G})$

   (b) $\text{curl}(\vec{F} + \vec{G}) = \text{curl}(\vec{F}) + \text{curl}(\vec{G})$

   (c) $\text{div}(\alpha\vec{F}) = \alpha\,\text{div}(\vec{F}) + \vec{F} \cdot \nabla\alpha$

   (d) $\text{curl}(\alpha\vec{F}) = \alpha\,\text{curl}(\vec{F}) + (\nabla\alpha) \times \vec{F}$

   (e) $\text{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot \text{curl}(\vec{F}) - \vec{F} \cdot \text{curl}(\vec{G})$

   (f) $\text{curl}(\text{curl}(\vec{F})) = \nabla(\text{div}\,\vec{F}) - \nabla^2\vec{F}$

7. Prove every continuous function $f(x,y,z)$ is the divergence of some vector field.