Change of Variables

Here are some examples computing integrals via change of variables.

1. Compute $\iint_R (x-3y)\ dA$ for R the triangle with vertices $(0,0)$, $(2,1)$, and $(1,2)$.

Solution: We solved this one in class.

2. Compute $\iint_R (4x+8y)\ dA$ for R the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$.

Solution: We solved this one in class.

3. Compute $\iint_R xy\ dA$ for R the region in the first quadrant bounded by the curves $y=x$, $y=3x$, $xy=1$, and $xy=3$. Use the transformation $x=\frac{u}{v}$, $y=v$.

Solution: Using the given transformation, we first seek to parametrize the given region in uv-coordinates. Notice $v=y\geq 0$ and $u=xv=xy\geq 0$ as R satisfies $x,y\geq 0$. Moreover, the defining equations for our region become the following after transformation:

$y=x$	becomes	$v=\frac{u}{v}$
$y=3x$	becomes	$v=3\cdot\frac{u}{v}$
$xy=1$	becomes	$\frac{u}{v}\cdot v=1$
$xy=3$	becomes	$\frac{u}{v}\cdot v=3$

Hence our new region is bounded by the curves $v^2=u$, $v^2=3u$, $u=1$, and $u=3$ in the first quadrant of the uv-plane. We may parametrize our region by

$$R_{uv}=\left\{(u,v):1\leq u\leq 3,\sqrt{u}\leq v\leq\sqrt{3u}\right\}.$$

Now the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)}=\operatorname{abs}\det\begin{bmatrix}v^{-1} & -uv^{-2}\\ 0 & 1\end{bmatrix}=\operatorname{abs}(v^{-1})=v^{-1}.$$

Hence we may integrate as follows to complete the computation.

$$\begin{aligned}\iint_R xy\ dA &= \int_{u=1}^{3}\int_{v=\sqrt{u}}^{\sqrt{3u}}\frac{u}{v}\cdot v\cdot v^{-1}\ dv\ du=\int_{u=1}^{3}\int_{v=\sqrt{u}}^{\sqrt{3u}}uv^{-1}\ dv\ du\\ &= \int_{u=1}^{3}u\cdot\Big[\ln(v)\Big]_{v=\sqrt{u}}^{\sqrt{3u}}\ du=\int_{u=1}^{3}u\cdot(\ln(\sqrt{3u})-\ln(\sqrt{u}))\ du\\ &= \tfrac{1}{2}\ln(3)\int_{u=1}^{3}u\ du=\tfrac{1}{4}\ln(3)\Big[u^2\Big]_{u=1}^{3}=2\ln(3)\end{aligned}$$

4. Compute $\iint_R \cos\left(\frac{y-x}{x+y}\right)\ dA$ for R the trapezoid with vertices $(0,1)$, $(0,2)$, $(1,0)$, and $(2,0)$.

Solution: The region R is naturally described in terms of lines $x+y=k$ and $y-x=k$ for constant k. In particular, we let $u=x+y$ and $v=y-x$. Solving for x and y in terms of u and v we obtain $x=\frac{1}{2}(u-v)$ and $y=\frac{1}{2}(u+v)$. Under this transformation we obtain

$$R_{uv}=\{(u,v):1\leq u\leq 2,-u\leq v\leq u\}.$$

On the other hand we compute the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)}=\operatorname{abs}\det\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}=\operatorname{abs}(\tfrac{1}{4}+\tfrac{1}{4})=\tfrac{1}{2}.$$

Hence we complete the computation as follows.

$$\begin{aligned}\iint_R \cos\left(\frac{y-x}{x+y}\right)\ dA &= \iint_{R_{uv}}\cos(vu^{-1})\tfrac{1}{2}\ dA_{uv}=\tfrac{1}{2}\int_{u=1}^{2}\int_{v=-u}^{u}\cos(vu^{-1})\ dv\ du\\ &=\tfrac{1}{2}\int_{u=1}^{2}\Big[u\sin(vu^{-1})\Big]_{v=-u}^{u}\ du=\tfrac{1}{2}\int_{u=1}^{2}(u\sin(1)-u\sin(-1))\ du\\ &=\tfrac{1}{2}\int_{u=1}^{2}2u\sin(1)\ du=\sin(1)\int_{u=1}^{2}u\ du\\ &=\sin(1)\cdot\tfrac{1}{2}\Big[u^2\Big]_{u=1}^{2}=\tfrac{3}{2}\sin(1)\end{aligned}$$

5. Compute $\iint_R \exp(x+y)\ dA$ for $R=\{(x,y):|x|+|y|\leq 1\}$.

Solution: We may describe the region R as follows using the piecewise definition of the absolute value.

$$R=\{(x,y):x+y\leq 1,x-y\leq 1,-x+y\leq 1,-x-y\leq 1\}$$

Thus our region is the region determined by the inequalities $-1\leq x+y\leq 1$ and $-1\leq x-y\leq 1$. Consider the transformation determined by $u=x+y$ and $v=x-y$; solving these equations for x and y in terms of u and v we obtain $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(u-v)$. On the other hand, the region R in uv-coordinates is precisely

$$R=\{(u,v):-1\leq u\leq 1,-1\leq v\leq 1\}.$$

Moreover the Jacobian of the transformation is precisely

$$\frac{\partial(x,y)}{\partial(u,v)}=\operatorname{abs}\det\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}=\operatorname{abs}(-\tfrac{1}{4}-\tfrac{1}{4})=\tfrac{1}{2}.$$

Hence we may integrate as follows to complete the computation.

$$\begin{aligned}\iint_R \exp(x+y)\ dA &= \int_{u=-1}^{1}\int_{v=-1}^{1}\exp\left(\tfrac{1}{2}(u+v)+\tfrac{1}{2}(u-v)\right)\cdot\tfrac{1}{2}\ dv\ du=\tfrac{1}{2}\int_{u=-1}^{1}\int_{v=-1}^{1}\exp(u)\ dv\ du\\ &=\tfrac{1}{2}\int_{u=-1}^{1}\exp(u)\Big[v\Big]_{v=-1}^{1}\ du=\int_{u=-1}^{1}\exp(u)\ du=\Big[\exp(u)\Big]_{u=-1}^{1}=e-e^{-1}\end{aligned}$$

6. Compute the volume of the solid ellipsoid $E = \left\{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$ where $a, b, c > 0$.

Solution: The transformation $x = au$, $y = bv$, $z = cw$ turns the ellipsoid E into a solid unit sphere

$$E_{uvw} = \left\{(u, v, w) : u^2 + v^2 + w^2 \leq 1\right\}.$$

Moreover the Jacobian of this transformation is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \operatorname{abs}\det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \operatorname{abs}(abc) = abc.$$

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Hence (recalling from class that the solid unit sphere has volume $\frac{4}{3}\pi$) we complete the computation.

$$\operatorname{vol}(E) = \iiint_E dV = \iiint_{E_{uvw}} abc \, dV_{uvw} = abc \iiint_{E_{uvw}} dV_{uvw} = \tfrac{4}{3}abc\pi$$